Survival Probability in the Life Annuity Insurance Model with Stochastic Return on Investments

Tatiana A. Belkina[1], Nadezhda B. Konyukhova[2], and Bohdan V. Slavko[3]

[1] Central Economics and Mathematics Institute of RAS, Moscow, Russia,
[2] Dorodnicyn Computing Center of RAS FRC CSC of RAS, Moscow, Russia
[3] National Research University – Higher School of Economics, Moscow, Russia
slavkobogdan@gmail.com

Abstract. We investigate the survival probability in the life annuity, or pension, insurance model when whole of the surplus (or a fixed its part) is invested into a risky asset with the price following the geometric Brownian motion. For the case of exponential distribution of revenue sizes, we formulate a singular boundary value problem for linear integro-differential equation and prove that the survival probability as a function of the initial surplus is the unique solution of this problem. Moreover, asymptotic representations for the survival probability both for small and large values of initial surplus are obtained. The efficient algorithm for the numerical calculation of the survival probability is described. Using computational experiments, we show that in the pension insurance business risky investments play a very important role in strengthening of the insurers solvency in a zone of small sizes of the surplus.

Keywords: life annuity insurance, dual risk model, survival probability, investment, risky asset, geometric Brownian motion, exponential distribution of revenue sizes

1 Introduction and Statement of the Problem

We consider the life annuity insurance model [9] (so called "dual risk model", see, e. g., [2]), where the surplus or equity of a company (in the absence of investments) is of the form

$$R_t = u - ct + \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0. \tag{1}$$

Here R_t is the surplus of a company at time $t \geq 0$; u is the initial surplus, c is the life annuity rate (or the pension payments per unit of time), assumed to be deterministic and fixed. $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ that, for any $t > 0$, determines the number of random revenues up to the time t; Z_k ($k = 1, 2, \dots$) are independent identically distributed random variables with a distribution function $F(z)$ ($F(0) = 0$, $\mathbf{E}Z_1 = m < \infty$) that determine the revenue sizes and are assumed to be independent of $N(t)$. These revenues arise at the moments of the death of policyholders.

In comparison with the classical non-life collective risk model (so called Cramér-Lundberg (CL) model, see [9]), the circumstances in (1) are reversed: in the classical model second and third summands have opposite signs. Since the "claims" in the dual model are negative (the jumps of the process (1) are positive), this model is also called the insurance model with *negative risk sums* or *compound Poisson model with negative claims* [3].

Let now the whole surplus be continuously invested into risky asset with price S_t following the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t,\ t \geq 0, \tag{2}$$

where μ is the expected return rate, σ is the volatility, B_t is a standard Brownian motion.

Then the resulting surplus process X_t is governed by the equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t + dR_t,\ t \geq 0, \tag{3}$$

with the initial condition $X_0 = u$, where R_t is defined by (1).

Remark 1. For the case when only fixed part of the surplus is invested into risky asset while the rest part is invested into a risk free asset with fixed return rate, the equation (3) is valued with modified parameters for the corresponding surplus process (see, e. g., [4]).

Denote $\varphi(u) = \mathbf{P}\left(X_t \geq 0,\ \ t \geq 0\right)$ the survival probability (i. e., the probability that bankruptcy will never happen); $\Psi(u) = 1 - \varphi(u)$ is the ruin probability. Let $\varphi_0(u) = \mathbf{P}\left(R_t \geq 0,\ \ t \geq 0\right)$ be the survival probability for the process (1) and $\Psi_0(u) = 1 - \varphi_0(u)$ be the corresponding ruin probability (in the absence of investments).

For the case of exponential distribution of revenue sizes, the following theorem is proved in [10].

Theorem 1. *Let $F(z) = 1 - e^{-z/m}$, $m > 0$, $\beta := 2\mu/\sigma^2 - 1$. Then:*
1) if $\beta > 0$ then $\Psi(u) = Ku^{-\beta}(1 + o(1))$, $\ u \to \infty$, for some $K > 0$;
2) if $\beta \leq 0$ then $\Psi(u) = 1$ for any u.

The formulation of this theorem is exactly the same as in [8], [14] for the non-life insurance model.

For the ruin probability $\Psi_0(u)$, it is easy to obtain an integro-differential equation (IDE) using the obvious modifications of the "differential argument" (see, e. g., [9]). In the case of exponential distribution of revenue sizes and if the safety loading is positive, i. e., the inequality $\lambda m > c$ is valid, this IDE has an exact solution satisfying boundary conditions $\lim_{u\to+0} \Psi_0(u) = 1$, $\lim_{u\to\infty} \Psi_0(u) = 0$; this solution has the form

$$\Psi_0(u) = \exp\left\{-(\lambda/c - 1/m)u\right\}. \tag{4}$$

Such slow decay as in Theorem 1 is in contrast with the exponential representation (4) in corresponding model without investment. This fact leads to the following conclusion, which has been made earlier for various models with non-heavy-tailed distributions of claims: investment of the whole surplus into risky assets in a zone of large values of the surplus impair the insurers solvency. The same conclusion remains true when only fixed part of the surplus is invested into risky assets (so called simple investment strategies), see [4] and references therein.

At the same time, the studies of optimal investment strategies, which maximize the survival probabilities in various settings of a problem for the CL model, show that the risky assets play a crucial role in strengthening of the insurer's solvency in a zone of small sizes of the surplus (see, e. g., [5] and references therein). The same conclusions concern the simple investment strategies for CL model and some its modifications, see [6].

The main goal of our paper is to identify the impact of simple investment strategies on the solvency in the dual risk model not only in the case of large surplus levels but also in the case of its small levels. For this purpose, we use the approach based on so called sufficiency theorems [4], which state that the solutions of singular problems for linear IDEs, generated by infinitesimal operators of the resulting surplus processes, define the corresponding survival probabilities. This approach is rather different from the one used in [10] and eliminates the need a priory to prove the twice continuously differentiability of the survival probability as well as the justification of boundary conditions. Solving the singular problem for IDE, we calculate the survival probability as a function of the initial surplus on the whole nonnegative semi-axis by means of the proposed algorithm; some results of numerical experiments are described. These results allow us to make conclusions about impact of the simple risky investment strategies on the solvency in the life annuity insurance model.

2 Some Preliminary Results

Recall at first that the infinitesimal generator $\mathcal{A}$ (see, e. g., [11]) of the process X_t defined by (3) has the form

$$(\mathcal{A}f)(u) = \frac{1}{2}\sigma^2 u^2 f''(u) + f'(u)(\mu u - c) - \lambda f(u) + \lambda \int_0^\infty f(u+z)\, dF(z), \quad (5)$$

for any function f from a certain subclass $\mathcal{D}$ of the space $\mathcal{C}^2(\mathbb{R}_+)$ of real-valued, twice continuously differentiable on $(0, \infty)$ functions.

In the assumption $\varphi(u) \in \mathcal{D}$, where $\varphi(u)$ is the survival probability of the process X_t, some considerations based on the generalized Ito's formula and the complete probability formula allow us to write the following equation for $u > 0$:

$$(\mathcal{A}\varphi)(u) = 0 \quad (6)$$

(see [10]; for the corresponding equation in the CL model with stochastic return on investment, see, e. g., [13]).

For the case of exponential distribution of revenue sizes, namely when

$$F(z) = 1 - \exp(-z/m), \quad m > 0, \tag{7}$$

the equation (6) has the form

$$\frac{1}{2}\sigma^2 u^2 \varphi''(u) + (\mu u - c)\varphi'(u) - \lambda\varphi(u) + \\ + \frac{\lambda}{m}\int_u^\infty \varphi(z)\exp(-(z-u)/m)dz = 0, \ u > 0. \tag{8}$$

Here we use the following transformation of non-Volterra integral operator in (5) with F of the form (7):

$$(J_m\varphi)(u) := \frac{1}{m}\int_0^\infty \varphi(u+z)\exp(-z/m)dz = \frac{1}{m}\int_u^\infty \varphi(z)\exp(-(z-u)/m)dz. \tag{9}$$

As noted in [10], the life annuity insurance case is rather different from the non-life insurance one because the change of two signs to the opposite ones in the equation defining the dynamics of the reserve leads to technical complications. In contrast to the non-life insurance case, the risk process (3) may leave the positive half-axis only in a continuous way. In [10] it was emphasized that the main difficulty in deriving the IDE is to prove the smoothness of the ruin probability. The other difficulty is to establish lower and upper asymptotic bounds for ruin probability in order to identify boundary conditions at infinity for the survival probability as the solution of the IDE. The smoothness of the ruin probability is studied in [10] using a method based on integral representations; as a tool for the proof of the asymptotic bounds for the ruin probability some theorems of the renewal theory are used.

As mentioned above in the introduction, in this paper we apply the approach based on sufficiency theorems (developed in [4] for the CL model and its modification with stochastic premiums; see also the earlier paper [13]) which allows us to avoid the a-priori proof of the the smoothness of the ruin probability as well as the justification of the boundary conditions at infinity.

For this purpose, we need a few preliminary propositions.

Lemma 1. *For the survival probability $\varphi(u)$ of the process* (3) *with the initial condition $X_0 = u$, the following relation is valid:*

$$\varphi(0) = 0, \tag{10}$$

i. e., the ruin occurs immediately at zero initial surplus.

The proof of this lemma is obvious due to the negativity of the deterministic component and we omit it. We provide below several statements concerning the properties of the solutions to IDE (8) satisfying the various conditions.

Lemma 2. *Let in IDE* (8) *the parameters* c*,* λ*,* σ*,* m *be fixed positive numbers and* μ *is arbitrary fixed number. Then if there exists a solution* $\varphi(u)$ *to IDE* (8) *with conditions*

$$\lim_{u\to+0}\varphi(u)=0, \quad \lim_{u\to\infty}\varphi(u)=1, \tag{11}$$

then this solution is unique.

This lemma can be easily proved by contradiction using the linearity of IDE.

To formulate some further auxiliary propositions, we will use also the following limiting conditions:

$$\lim_{u\to+0}|\varphi'(u)|<\infty, \quad \lim_{u\to+0}u\varphi''(u)=0. \tag{12}$$

$$\lim_{u\to\infty}u\varphi'(u)=0, \quad \lim_{u\to\infty}u^2\varphi''(u)=0. \tag{13}$$

Lemma 3. *Let in IDE* (8) *the parameters* c*,* λ*,* σ*,* m *be fixed positive numbers and* μ *is arbitrary fixed number. If there exists the solution* $\varphi(u)$ *of IDE* (8)*, satisfying conditions* (11), (12)*, then*

$$0\le\varphi(u)\le 1, \quad u\in\mathbb{R}_+; \tag{14}$$

moreover,

$$0<\lim_{u\to+0}\varphi'(u)<\infty. \tag{15}$$

The first part of this statement may be proved by contradiction (see the proof of a similar assertion in [6]). Let us prove the second part of the statement. Indeed, from the IDE (8) and conditions (11) and (12) we have the relation

$$-c\varphi'(+0)+\frac{\lambda}{m}\int_0^\infty\varphi(z)\exp\left(-z/m\right)dz=0, \tag{16}$$

wherefrom, taking into account the proved above relation (14), the conditions (11) and positiveness of c and other parameters, we conclude that (15) is valid.

The following lemma is essential auxiliary statement for further study of the initial problem.

Lemma 4. *Let in IDE* (8) *all the parameters* c*,* λ*,* μ σ*,* m *be fixed positive numbers. Then the singular IDE problem* (8), (11)–(13) *is equivalent to the singular problem for ODE*

$$\frac{1}{2}\sigma^2u^2\varphi'''(u)+\left(\mu u+\sigma^2u-c-\frac{1}{2m}\sigma^2u^2\right)\varphi''(u)+ \\ +\left(\mu-\lambda-\frac{\mu u-c}{m}\right)\varphi'(u)=0, \tag{17}$$

defined on $\mathbb{R}_+$*, with conditions* (11)–(13).

Proof. Let $\varphi(u)$ be satisfying IDE (8). Let us show that it satisfies also ODE (17). It is easy to check that, for the operator (9), the following relation is valid:

$$\frac{d}{du}(J_m\varphi)(u) = \frac{1}{m}(J_m\varphi)(u) - \frac{1}{m}\varphi(u). \tag{18}$$

By differentiating IDE (8) and taking into account the relation (18) we obtain

$$\frac{1}{2}\sigma^2u^2\varphi'''(u) + \sigma^2u\varphi''(u) + (\mu u - c)\varphi''(u) + \\ + (\mu - \lambda)\varphi'(u) + \frac{\lambda}{m}(J_m\varphi)(u) - \frac{\lambda}{m}\varphi(u) = 0. \tag{19}$$

The obvious linear combination of IDEs (8) and (19), which exclude the integral term $(J_m\varphi)(u)$, leads to ODE (17).

Conversely, let now $\hat{\varphi}(u)$ be satisfying ODE (17) and conditions (11) and (13). Let us show that it satisfies also the IDE (8). Denote $g(u)$ the left-hand side of the equation (8) with function $\hat{\varphi}(u)$. Then

$$g(u) = -\lambda\hat{\varphi}(u) + \lambda(J_m\hat{\varphi})(u) + (\mu u - c)\hat{\varphi}'(u) + \frac{1}{2}\sigma^2u^2\hat{\varphi}''(u),$$

$$g'(u) = \frac{\lambda}{m}[(J_m\hat{\varphi})(u) - \hat{\varphi}(u)] + (\mu - \lambda)\hat{\varphi}'(u) + \\ + (\mu u + \sigma^2u - c)\hat{\varphi}''(u) + \frac{1}{2}\sigma^2u^2\hat{\varphi}'''(u).$$

Hence,

$$g'(u) - \frac{g(u)}{m} = \left(\mu - \lambda - \frac{\mu u - c}{m}\right)\hat{\varphi}'(u) + \\ + \left(\mu u + \sigma^2u - c - \frac{1}{2m}\sigma^2u^2\right)\hat{\varphi}''(u) + \frac{1}{2}\sigma^2u^2\hat{\varphi}'''(u),$$

and, in view of the fact that the function $\hat{\varphi}(u)$ is a solution of ODE (17), we obtain

$$g'(u) - \frac{g(u)}{m} = 0. \tag{20}$$

The solution of ODE (20) has the form

$$g(u) = C\exp(u/m), \quad u > 0, \tag{21}$$

where C is an arbitrary constant. It is easy to see that for the function $\hat{\varphi}(u)$, which satisfies conditions (11) and (13), the following relation is valid:

$$\lim_{u\to+\infty}\frac{1}{m}\int_u^{\infty}\hat{\varphi}(s)\exp(-(s-u)/m)ds = 1. \tag{22}$$

Then, taking into account the definition of $g(u)$, equality (22) and conditions (11), (13) we conclude that the equality $\lim_{u\to\infty} g(u) = 0$ holds. Consequently, in view of positiveness m, the constant C in (21) should be equal to zero for this solution, i. e., $g(u) \equiv 0$. Thus, $\hat{\varphi}(u)$ is the solution of IDE (8).

It remains to note that the whole set of conditions is the same for the two considered problems for IDE and ODE, and lemma is proved.

To establish a connection between the original problem of the survival probability investigation and a singular problem for IDE, we need also the following statement which we call the sufficiency theorem [4].

Theorem 2. *Let in IDE* (8) *all the parameters be positive numbers and the inequality*

$$2\mu > \sigma^2 \tag{23}$$

be fulfilled. Suppose IDE (8) *has a twice continuously differentiable on* $(0, \infty)$ *solution* $\varphi(u)$ *subject to conditions*

$$0 \leq \varphi(u) \leq 1, \quad u \in \mathbb{R}_+, \tag{24}$$

$$\lim_{u\to+\infty} \varphi(u) = 1. \tag{25}$$

Then $\varphi(u)$ *is the survival probability for the process* (3) *with initial state* $X_0 = u$.

The proof of this theorem is completely analogous to the proof of Theorem 3.1 in [4].

3 Main Theorem

For the considered case of the exponential distribution of revenue sizes, we establish the following statement.

Theorem 3. *Let* $F(z)$ *be of the form* (7), *all the parameters* μ, σ^2, m, c, λ *be fixed positive constants, and let the condition* (23) *be satisfied. Then the following assertions hold:*

(I) *the survival probability* $\varphi(u)$ *of the process* (3) *with initial condition* $X_0 = u$ *is the solution to the singular boundary value IDE problem* (8), (11);

(II) *this solution is unique and satisfies the following relations:*

$$0 \leq \varphi(u) \leq 1, \quad u \in \mathbb{R}_+, \tag{26}$$

$$0 < \lim_{u\to+0} \varphi'(u) < \infty; \tag{27}$$

(III) *the survival probability* $\varphi(u)$ *may be calculated by the formula*

$$\varphi(u) = 1 - \int_u^{\infty} \psi(s) ds, \tag{28}$$

where $\psi(u) = \varphi'(u)$ is the solution of the following singular problem for ODE:

$$\frac{1}{2}\sigma^2 u^2 \psi''(u) + \left(\mu u + \sigma^2 u - c - \frac{1}{2m}\sigma^2 u^2\right)\psi'(u) + \\ + \left(\mu - \lambda - \frac{\mu u - c}{m}\right)\psi(u) = 0, \quad (29)$$

$$0 < u < \infty,$$

$$\lim_{u \to +0} |\psi(u)| < \infty, \quad \lim_{u \to +0} u\psi'(u) = 0, \quad (30)$$

$$\lim_{u \to \infty} u\psi(u) = 0, \quad \lim_{u \to \infty} u^2\psi'(u) = 0; \quad (31)$$

$$\int_0^\infty \psi(s)ds = 1; \quad (32)$$

(IV) $\varphi(u)$ *has the asymptotic representations*

$$\varphi(u) \sim D_1\left(u + \sum_{k=2}^{\infty} D_k u^k/k\right), \quad u \sim +0, \quad (33)$$

where

$$D_1 = \varphi'(+0),$$
$$D_2 = (\mu - \lambda + c/m)/c, \quad (34)$$
$$D_3 = \left[D_2(2\mu + \sigma^2 - \lambda + c/m) - \mu/m\right]/(2c), \quad (35)$$
$$D_{k+1} = [D_k(k(k-1)\sigma^2/2 + \mu k - \lambda + c/m) - \\ - D_{k-1}((k-2)\sigma^2/(2m) + \mu/m)]/(kc), \quad k = 3, 4, \ldots, \quad (36)$$

and

$$\varphi(u) = 1 - Ku^{1-2\mu/\sigma^2}(1 + o(1)), \quad u \to \infty, \quad (37)$$

where $K > 0$ is a constant;

(V) *as $u \to +0$, the behavior of the survival probability derivatives depends on the relations between the parameters, in particular on a sign of the coefficient $i_{\mathrm{r}} = (\lambda - \mu)m - c$: (1) if $i_r \geq 0$, then $\lim_{u\to+0}\varphi''(u) \leq 0$, moreover, the solution φ is concave on $\mathbb{R}_+$; (2) if $i_r < 0$, then $\lim_{u\to+0}\varphi''(u) > 0$, the solution φ is convex in a some neighborhood of zero and has an inflexion point.*

Sketch of the proof. At first, we need to establish the existence and uniqueness of the solution to the problem (29)–(32) and to study its asymptotic behaviors for large and small values of u. For this purpose, we have to investigate the singular problems (29), (30) and (29), (31) separately, taking into account that ODE (28) has irregular singular points at zero and infinity (about singular points for ODEs see, e. g. [15]).

By using methods of the investigation of ODEs with singular points [15], [12] we obtain asymptotic representation for families of solutions to this singular problems (see also [6] and references therein for analogous investigation with application of these methods for CL model with investment in details). As result we have that ODE (29) for small $u > 0$ has a two-parameter family of solutions $\psi(u, D_1, C_1)$ and for these solutions the following asymptotic representation holds:

$$\psi(u, D_1, C_1) =$$
$$= D_1\,(1 + \psi_1(u)) + C_1 \exp\left(-2c/(\sigma^2 u)\right) u^{-2\mu/\sigma^2}(1 + \psi_2(u)), \quad u \to 0. \quad (38)$$

Here C_1 is a parameter, $\psi_2(u) = o(1)$, $u \to +0$, the function $\psi_1(u)$ can be represented by asymptotic series

$$\psi_1(u) \sim \sum_{k=1}^{\infty} D_{k+1} u^k, \quad u \sim +0, \quad (39)$$

where the coefficients D_k, $k = 2, 3, \dots$, may be found from the recurrence relations (34)–(36).

It is obvious that the conditions (30) hold for all the solutions of the family $\psi(u, D_1, C_1)$, i. e., for all the solutions to ODE (29).

Under condition (23), ODE (29) has a one-parameter family of solutions $\psi(u, C_2)$ which are integrable at infinity. For these solutions, the following asymptotic representation holds as $u \to \infty$:

$$\psi(u, C_2) = C_2 u^{-2\mu/\sigma^2}(1 + o(1)), \quad \psi'(u, C_2) = -\frac{2\mu}{\sigma^2 u} C_2 u^{-2\mu/\sigma^2}(1 + o(1)). \quad (40)$$

It is obvious that the conditions (31) hold for all solutions of this family (i. e., for all integrable at infinity solutions of (29)) iff the condition (23) is fulfilled.

Thus, if the condition (23) is fulfilled, then there exists one-parameter family of solutions to the problem (29)-(31). All the solutions of the equation (29) are bounded at zero, and all bounded and integrable at infinity solutions belong to this family and have the asymptotic representations (38) and (40). The condition (32) extracts the unique solution from this family. It is clear that this solution satisfies the conditions (30) and (31). Then, taking into account Lemma 4, it is easy to see that the function $\varphi(u)$ defined by the formula (28) is the solution to the IDE problem (8), (11)–(13) and has the asymptotic representations (33) and (37). Therefore, according to Lemma 3, the relations (14) also hold and, in view of Theorem 2, for any $u \in \mathbb{R}_+$, the value of $\varphi(u)$ is the survival probability for the process (3) with initial state $X_0 = u$. In accordance to Lemma 2 (about the uniqueness) this probability as a function of u satisfies conditions (12) and (13) with necessity. In view of Lemma 3, we have also that the inequalities (15) take place, and, in accordance to the asymptotic representation (33), we conclude that $\varphi''(+0)$ and the expression $\mu - \lambda + c/m$ are of the same sign. The sketch of the proof is completed.

4 Numerical Examples

The studies given in previous sections allow us to suggest computationally simple and theoretically justified algorithm for numerical calculation of the survival probability in the considered model. This algorithm requires to solve the singular Cauchy problem from infinity (29), (31) with the normalizing condition (32). Then it remains to use the relation (28) (recall that all the solutions of ODE (29) are bounded as $u \to +0$ and all the integrable at infinity solutions form the one-parameter family). To solve numerically the problem (29), (31) we realize previously the equivalent transfer of the limit conditions (31) from infinity to a large finite point using the results [7], [12]. For the first approximation, such approach yields boundary condition at a finite point $u = u_\infty \gg 1$ as follows

$$\psi'(u_\infty) \approx -\frac{2\mu}{\sigma^2 u_\infty}\,\psi(u_\infty)$$

(the same relation follows also from (40)).

For general ODE systems with pole-type singular points, the theory of boundary condition transfer from singular points is developed; such transfer can be realized by construction of the stable initial manifolds, or the Lyapunov manifolds of conventional stability, at the neighborhoods of singular points (see, e. g., [1] and references therein). On the application of such approach in actuarial mathematics, see [6] and references therein.

Numerical experiments (see in particular Figs. 1, 2) show that risky investments improves survival probability at a zone of small values of initial surplus in the case of positive safety loading (Fig. 2). Moreover, risky investments make survival possible in the case of negative safety loading, see Fig. 1 (in this case survival is impossible without investments).

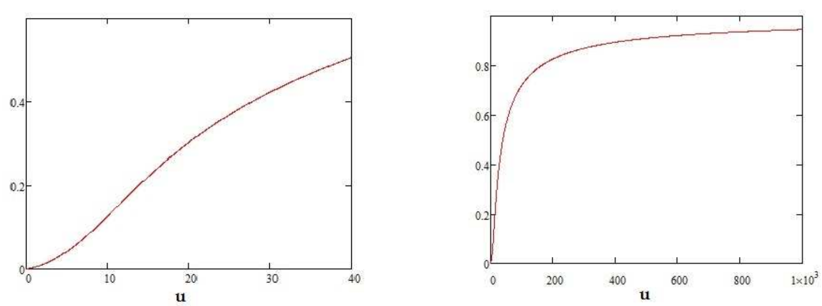

Fig. 1. The graphs of the survival probability as a function of initial surplus in the case of negative safety loading for two different scales ($\lambda = 1$, $m = 2$, $c = 4$, $\mu = 0.2$, $\sigma^2 = 0.23$).

5 Conclusions

To study the impact of investments with stochastic return on survival probability in the life annuity insurance model we use the approach based on so called

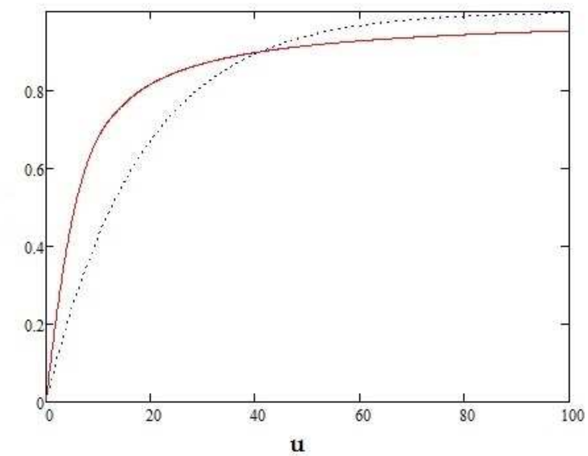

Fig. 2. Survival probability as a function of initial surplus in the case of positive safety loading ($\lambda = 1$, $m = 2$, $c = 1.8$) both with risky investments (solid curve, $\mu = 0.2$, $\sigma^2 = 0.22$) and without investments (dotted curve).

sufficiency theorem and the existence theorems for the corresponding singular problems for IDEs (see [4]). This unified approach eliminates need to proof regularity of the survival probability as well as to use its upper and lower bounds. Moreover, the solving of above singular problem for IDE leads to calculation of the survival probability on all the non-negative semi-axis. We reduce the problem (8), (11) to a certain initial problem from infinity for some second order ODE with respect to the derivative of the survival probability with normalizing condition. As a result of calculations, we conclude in particular that if the value of safety loading ($\lambda m - c$) in the model (1) is negative or sufficiently small and the surplus is small too, then the use of the risky investments allows to increase the survival probability significantly.

References

1. Abramov, A.A., Konyukhova, N.B.: Transfer of admissible boundary conditions from a singular point for systems of linear ordinary differential equations. Sov. J. Numer. Anal. Math. Modelling (VNU Science Press, Utrecht, The Netherlands) 1(4), 245–265 (1986)
2. Albrecher, H., Badescu, A., Landriault, D.: On the dual risk model with tax payments. Insurance Math. Econom. 42, 1086–1094 (2008)
3. Asmussen, S., Albrecher, H.: Ruin Probabilities. Advanced Series on Statistical Science and Applied Probability, V.14. World Scientific, Singapore (2010)
4. Belkina, T.A.: Risky investment for insurers and sufficiency theorems for the survival probability. Markov Processes Relat. Fields. 20, 505–525 (2014)
5. Belkina, T., Hipp, C., Luo, S., Taksar, M.: Optimal constrained investment in the Cramer-Lundberg model. Scandinavian Actuarial Journal 5, 383–404 (2014)

6. Belkina, T.A., Konyukhova, N.B., Kurochkin, S.V.: Dynamical insurance models with investment: Constraint singular problems for integro-differential equations. Comput. Math. Math. Phys. 56(1), 47-98 (2016)
7. Birger, E.S., Lyalikova, N.B.: Discovery of the solutions of certain systems of differential equations with a given condition at infinity, I. U.S.S.R. Comput. Maths. Math. Phys. 5(6), 1–17 (1965); On finding the solutions for a given condition at infinity of certain systems of ordinary differential equations, II. U.S.S.R. Comput. Maths. Math. Phys. 6(3), 47–57 (1966)
8. Frolova, A., Kabanov, Yu., Pergamenshchikov, S.: In the insurance business risky investments are dangerous. Finance Stochast. 6(2), 227-235 (2002)
9. Grandell, J.: Aspects of Risk Theory. Springer, Berlin–New York (1991)
10. Kabanov, Yu., Pergamenshchikov, S.: In the insurance business risky investments are dangerous: the case of negative risk sums. Finance Stochast. 20(2), 355–379 (2016)
11. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus. Springer-Verlag, NY (1988)
12. Konyukhova, N.B.: Singular Cauchy problems for systems of ordinary differential equations. U.S.S.R. Comput. Maths Math. Phys. 23(3), 72–82 (1983)
13. Paulsen, J., Gjessing, H.K.: Ruin theory with stochastic return on investments. Adv. Appl. Probab. 29(4), 965–985 (1997)
14. Pergamenshchikov, S. and Zeitouny, O.: Ruin probability in the presence of risky investments. Stochastic Process. Appl. 116(2), 267–278 (2006)
15. Wasow, W.: Asymptotic Expansions for Ordinary Differential Equations. Dover, New York (1987).