Investigating Representational Dynamics in Problem Solving

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Abstract

Successful problem solving relies on the availability of suitable mental representations of the task domain. In more complex, and potentially ill-defined problems, there might be a wide variety of representations to choose from and it might even be beneficial to change them during problem solving. To explore such dynamics on the representational level, we developed a complex spatial transformation and problem solving task. In this task, subjects are asked to repeatedly mentally cross-fold a sheet of paper, and to predict the resulting sheet geometry. Through its deliberate under-specification and difficulty, this task requires subjects to find new and better fitting representations – ranging from visuospatial imagery to symbolic notions. We present an overview of the task domain and discuss various ways of representing the task as well as potential dynamics between them.

1 Introduction

Often, the difficulties of problem solving lie not only in how to perform heuristic search, but start with how to *understand* a given task [Van88]. In the study of problem solving, task understanding is typically conceptualised as "setting up" one's mental representation of the problem – its goals, constituents and possible operations – and considered a preparatory phase before the actual problem solving activity ensues in a subsequent solution phase [SH76, Vos06]. However, there is empirical evidence which suggests considerable interaction between task understanding and problem solving. For example, evidence suggests that pertinent phenomena such as insight, analogy, and transfer can be explained best in terms of changes of one's representation *during* problem solving [GW00, KOHR99, KRHM12]. Furthermore, developmental studies have shown that the use of different solution strategies, potentially employing distinct problem representations, might "overlap" during problem solving [Sie02, Sie06]. This suggests that *representational dynamics*, i. e. ongoing changes to how one represents a given task, might play an essential role throughout problem solving.

However, to date systematic investigations of representational dynamics, mapping out problem solving activity on the "representational level", are still missing. Given the predominant focus on the research of heuristic search, tasks are usually designed to constrain subjects to well-defined problem spaces [Goe10]. Considering how closely

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such problem spaces resemble their task environment, representational dynamics are thus already precluded by task design and presentation. In contrast, we propose the investigation of tasks that elicit representational dynamics as a regular part of problem solving. In order to make progress in solving such a task, subjects are continuously challenged to acquire more knowledge about – and potentially change their perspective towards – the task domain.

To this end, we present a complex spatial transformation and problem solving task in the domain of iterated paper folding. In the following, our focus lies on a first description of the task domain (Sect. 3). With this, we establish a prerequisite for systematic representational-level analyses of problem solving in the targeted domain. As a first step towards such analyses, we conclude with a discussion of various ways of representing the task mentally and the potential dynamics between those (Sect. 4), and a few final remarks discussing further aspects of this task which are beyond the scope of this introductory paper (Sect. 5).

2 Task: Iterated Paper Folding

The task we want to discuss in this paper consists of two subtasks which are presented consecutively, i. e. only after completion of the first subtask the second one is revealed:

1. Imagine cross-folding a sheet of paper and inspect the folded sheet,
2. Draw 2D sketches of the forms of edges on each side of the folded paper.

In subsequent iterations of the task, the number of times the sheet is to be cross-folded (and its sides to be sketched) is *incremented* – resulting in multiply-folded sheets ("folds"). Hence, while the subtasks stay the same in all iterations, the complexity of the folds to be made is increased with each iteration.

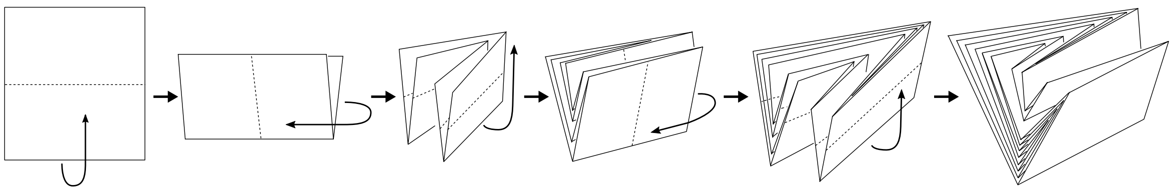

Figure 1: Cross-folding a sheet of paper five times (by alternating between perpendicular folding directions).

This task has several characteristics that make representational dynamics an essential part of human subjects solving it successfully[1]. As a prerequisite for this, it is important to note that the domain of iterated paper folding is such that problem solvers will find a variety of generally distinct representations effective, each with their merits and weaknesses. For instance, the domain can be approached visuospatially (as a case of mental imagery) or analytically (as case of spatial or even symbolic reasoning), but also more intricate combinations of these approaches are feasible (see Sect. 4). Moreover, depending on one's prior knowledge, and the experience gathered in successive iterations, one's conceptual understanding of the domain's underlying principles (see Sect. 3) permits the development of increasingly efficient representations. Taken together, these characteristics provide ample opportunity for representational dynamics.

In the initial engagement with the task, the noted variety of representational possibilities is likely to be most apparent, since the task is posed as an *ill-defined problem* [Ree15, Sim73, Rei65]: Besides the instructions for the first subtask being deliberately vague, no further verbal or graphical hints are provided. Thereby the task instructions do neither "prescribe" the use of specific representations or procedures, nor do they state a precise goal. To form a productive understanding of the task, subjects have to explore the task domain, potentially considering different kinds of more or less suitable representations and procedures [BV02, Goe10]. At the outset, problem solving activity will thus likely be relatively idiosyncratic, taking place in incomplete or incoherent, and dynamically changing problem spaces based on the subject's prior experiences with and familiar knowledge about paper folding.

While at later stages problem solving activity should become substantially more coherent, the reason to still expect sustained dynamics lies with the task's *iterative procedure*. Since with each iteration the complexity of the folds increases substantially, generating them gets progressively more challenging. Thus, simply applying the

[1]Presuming that from the start one would apply a form of representation that is well-suited for the complexity of later iterations, and assuming the availability of unlimited processing capacities, it would theoretically be possible to solve our task without changes in how it is represented – but both of these are not the case for typical human subjects (owing to both, the capacity limits of mental imagery, and a lack of detailed knowledge about the consequences of iterated folding).

representations and procedures developed so far to the new challenges will not suffice. To counter these increasing cognitive demands, subjects will instead have to find more efficient ways of representing and manipulating cross-folds – furthering their understanding of the task domain in the process, and effectively changing the problem spaces they are operating in.

Related work

While the task presented has been newly developed and there is to our knowledge no work directly concerning iterated cross-folding, there is a vast amount of work on other types of paper folding. For instance, there are psychological investigations of the mental folding of cube nets, and unfolding of mutliply-folded sheets with holes punched into them [HHPN13, SF72]. There is also work on the qualitative modelling of these kinds of tasks [Fal16], as well as mathematical and algorithmic descriptions of more complex paper folding such as Origami [DO07, IGT15]. Particularly noteable with respect to our work is a study of origami folding tasks which investigated how people reconceptualise the task and its constituents over the course of the study, thus equally emphasising possibilities for conceptual and representational change [TT15].

3 Overview of the Task Domain

In the following we provide a systematic overview of the domain of iterated paper folding in terms of its task-relevant entities and their regular interrelation, which subjects might represent in some form or another when engaging with the task. With this, we establish an objective point of reference for conducting and discussing representational-level analyses.

The overview is divided in three parts: First, Sect. 3.1 explains the *procedural details* of how to fold (illustrating the extent of under-specification in the first subtask). Then, Sect. 3.2 describes the *forms* the sheet's sides assume after being folded, and which are closely related to the sketches asked to be drawn for the second subtask. Finally, Sect. 3.3 describes the *relations* holding between these sides and between consecutive folds, respectively.

3.1 Folding Procedures

Cross-folding can be defined as *folding a sheet such that it is halved in middle by each fold and such that consecutive creases are perpendicular to each other.* Ignoring the sheet's size, thickness, and aspect ratio – the n-th cross-fold F_n is uniquely determined by:

(a) the initial spatial orientation of F_0
(b) the number of times folded (n)
(c) the folding procedure used

A folding procedure is determined by how exactly the sheet is being folded, most importantly the *direction* in which the fold is made. Initially (for $F_0 \to F_1$), there are 8 directions in which we can fold, two of which yielding identical folds (Fig. 2). Since cross-folding requires successive creases to be perpendicular, from F_1 onwards there are only 4 directions that can be chosen.[2]

When combining folding directions arbitrarily, there are 2^{2n+1} different folding procedures for $F_0 \to F_n$. Of special interest, however, are procedures in which folding directions are combined *systematically*, such as alternating between the same two perpendicular directions (e. g. Fig. 1). Using such a procedure limits the number of possibilities to 32 (8 directions $\times$ 4 perpendicular directions).

Alternatively, a cross-fold can also be achieved by introducing a 90°-rotation-step between two foldings in the same direction (e. g. Fig. 3). For such a procedure, there are only 16 possibilities (8 folding directions $\times$ 2 rotation directions).

Even though all of these procedures yield a cross-fold and hence fulfil the first sub-task equally, it is important to distinguish between them. Since the cross-folds produced by them differ in certain respects, they are relevant when trying to identify subjects' representations and evaluating their performance in the second subtask.

[2]Whereas folding in one of the other 4 directions would yield a *parallel fold*.

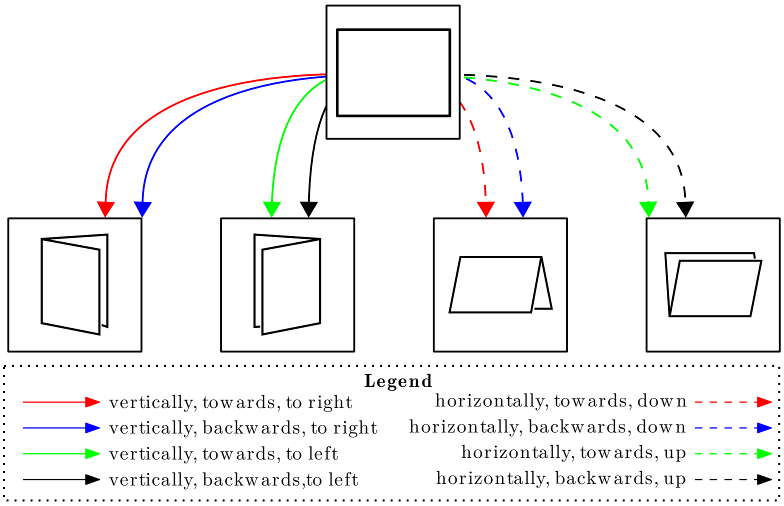

Figure 2: Eight possibilities of turning F_0 into F_1.

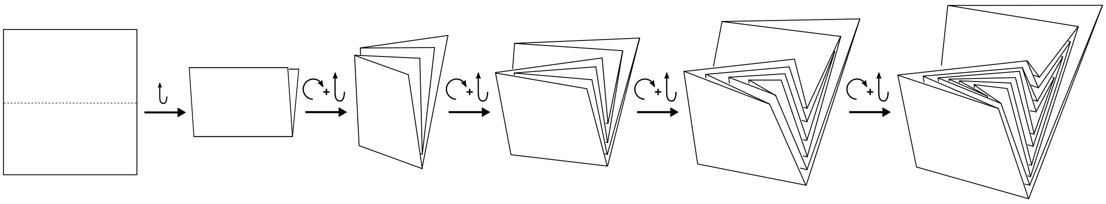

Figure 3: A folding procedure which rotates the sheet 90°clockwise before folding horizontally-towards-up (rotation steps not depicted)

3.2 Fold Forms

Describing the consequences of folding on the sheet, we can identify the occurrence of regular schematic forms, which all cross-folds have in common.

Depicting each side of a cross-fold two-dimensionally (as required by the second subtask) five *basic forms* can be observed. They are the forms of all folds from F_2 onwards, namely: Two uncreased rectangles, the crease itself (I), a single creased edge (V), a double V (DV), and a nested V (NV).

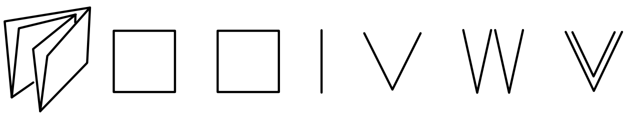

Figure 4: An exemplary illustration of F_2, its six sides, and their positions: rectangles (front, back), I (right), V (bottom), DV (left), and NV (top).

The rectangles, I, and V (re-)appear unchanged in all folds, but DV and NV only retain their basic shapes, with every $DV_{\geq 2}$ consisting of two sub-figures side-by-side, and every $NV_{\geq 2}$ consisting of two nested sub-figures. As the number of folds increase, they show more complex creasing of edges, hence they are referred to as a fold's two *complex sides.*[3]

Figure 5: NV_0 to NV_5 (left to right), as produced by the procedure shown in Fig. 1.

Looking at F_1 and F_2, it might seem that while there are many different folding procedures (see Sect. 3.1),

[3]Hence, the remaining description of our task domain will be mainly concerned with those two sides.

they always bring forth the exact same folds, only in different spatial orientations. Yet, with higher fold numbers certain differences start to appear.

3.2.1 Left-handed & Right-handed Folds

Beginning with F_3 folds are *chiral*, i. e. they are no longer identical with their mirror images and we have to distinguish left- and right-handed variants. For instance, the folding procedure shown in Figures 1 and 5 produces folds of alternating chirality.

Figure 6: The achiral NV_2 next to left- and right-handed variants of NV_3.

3.2.2 In-folding & Out-folding

Starting with F_4, different folding directions produce yet another distinction: Depending on whether a new fold F_n encompasses the V_{n-1} with the NV_{n-1} or the other way around, we distinguish *out-* and *in-folds* (denoted by a superscript I and O). Folding procedures with fixed direction and rotation will always yield out-folds, whereas non-rotating or alternately-rotating procedures can produce both kinds.

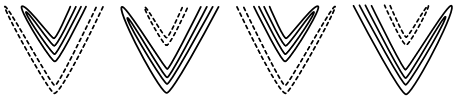

Figure 7: From left to right: Left-handed NV_4^O, NV_4^I, right-handed NV_4^O, NV_4^I.

3.2.3 Mixed Folds

Furthermore, depending on the folding direction either an in- or an out-fold is produced each time we fold. This means that if allowing arbitrary combinations of folding directions, folds cannot only be "purely" in- or out-folded, but they can be *mixed folds*, with an ever increasing number of possible combinations of in- and out-folding. This results in a total number of 2^{n+1} fold variants for F_n (as opposed to an upper bound of 32 variants if we disallowed arbitrary folding directions, and hence mixed folds).

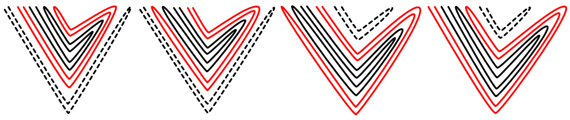

Figure 8: Four different NV_5, from left to right: Out-fold, mixed (in-folded out-fold), mixed (out-folded in-fold), in-fold.

3.3 Regular Relations

While in theory, every detail of how a fold will look like and how its sides relate to each other can be derived from the chosen folding procedure, these details can be relatively hard to see. Hence, in the following section, we describe the most important of these relations holding in the general case.

3.3.1 Within-fold Regularities

While folds can vary in their orientation, and in the detailed structure of their sides etc., the spatial relations holding between a fold's individual sides are the same for all cross-folds (see Table 1).

Table 1: An overview of the spatial relations between the sides of a fold.

	V	DV	NV
I	$\perp$	$\|$	$\perp$
V		$\perp$	$\|$
DV			$\perp$

($\|$ = spatially opposite, $\perp$ = perpendicularly adjacent)

3.3.2 Between-fold Regularities

In an equally general manner, we can describe the relations holding between the four non-rectangular sides of two consecutive folds F_n and F_{n+1} (for $n \geq 2$). Table 2 presents these relations as rules of how to manipulate each side of F_n two-dimensionally in order to generate F_{n+1}'s sides from them.

Table 2: An overview of the generative relations between the sides of consecutive folds.

I_{n+1}:	V_{n+1}:	DV_{n+1}:	NV_{n+1}:
new	fold(I_n)	align(V_n, NV_n)	fold(DV_n)

3.4 Summary

While there are many more advanced facts about the task domain[4], above we have introduced the domain's basic properties: (a) The different procedures that can be used to make cross-folds, (b) the forms a sheet's sides assumes after being cross-folded in different ways, and (c) the regular relations holding between a fold's sides.

Following from these basic properties, the large number of possible cross-folding procedures, with its intricate distinctions in higher iterations (chirality, in-/out-folding etc.), as well as the overall increasing complexity of higher folds, present particular challenges to subjects which will potentially give rise to representational dynamics. Additionally, the distinctions presented here are also relevant when trying to identify the representations subjects might use in solving the task.

4 Discussion: Representational Dynamics

In the following, we discuss which representational dynamics can be expected in the task. We present how people have been shown to solve mental folding tasks in general, using different varieties of representations. On this basis, we point out potential representational dynamics that can ensue within and between these kinds of representations in the domain of iterated paper folding.

Generally, research on spatial transformation distinguishes between two kinds of solution approaches which presume largely different kinds of representations. As indicated earlier, there are *visuospatial approaches*, i. e. imagining a 3D object, transforming it, and then "seeing" the result, and *analytic approaches*, i. e. understanding and solving the problem based on explicit domain knowledge [HHPN13]. Furthermore, visuospatial and analytic approaches are usually conceived as the poles of a continuum, allowing for *mixed approaches* [GF03].

When faced with the repeated challenges of the task, subjects will have to change their perspectives towards the problem several times, thus furthering their understanding of it, and taking advantage of the merits of different approaches. It is thus necessary to discuss each approach in some detail.

4.1 Visuospatial Approaches

In terms of a visuospatial approach, the mental imagery of paper folding is typically conceived of as the mental analogue of folding physically [SF72]. The physical process of folding is a non-rigid transformation, i. e. folding an object affects its individual parts differently [HHPN13]. In cross-folding, for instance, a single act of folding affects a sheet's sides in distinct ways (cf. Table 2). Consequently, in order to mentally form a visuospatial representation of a fold as whole, subjects will likely require multiple repetitions of the same transformation of a sheet, while variably attending to its individual sides.

[4] For instance, some readers might be interested that the number of nestings in a fold's NV_n corresponds to the partial sum $\sum_{i=0}^{n-1} 2^{\lfloor \frac{i}{2} \rfloor}$ (sequence A027383 in the OEIS [OEI17]).

The transformation of a visuospatial representation is usually understood as akin to sensorimotor transformations, i. e. an analog transformation of one visuospatial representation into another one mediated by a motor process [Iac11, MK09]. While this transformation is described as analogous to the physical act of folding, visuospatial representations often already leave out many physical details which are irrelevant to the task (such as aspect ratio or size). Crucially, in such a process information on the parts and their spatial interrelation is merely implicit. Presuming a certain everyday familiarity with the activity of cross-folding, visuospatial approaches have the advantage that subjects can perform them without much explicit knowledge about the task domain. Their effectiveness, however, is limited to spatially rather simple or very familiar complex objects [BFS88].

Consequently, when attempting to mentally cross-fold for the first few times, it might be possible to fold F_1 and F_2 in this manner. However, with increasing complexity of the folds, even attending to their visuospatial representations side-by-side will eventually become too demanding, and subjects have to find other approaches. Consider for example Fig. 9, where a DV_2 is being transformed into an NV_3 (as part of folding F_3 from F_2). Depending on one's familiarity with cross-folding, this might already be a very advanced transformation and at the boundary of what a typical subject is able to achieve in a purely visuospatial manner.

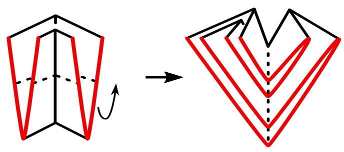

Figure 9: $F_2 \to F_3$, attending to the transformation of DV_2 into NV_3.

4.2 Analytic Approaches

Analytic approaches to mental folding are more straightforward cases of *problem solving* or *spatial reasoning*. To this end, the mental object is construed as a symbolic representation of interrelated parts. As opposed to visuospatial approaches, spatial information is not implicit in such representations, but has to be made explicit as a semantic relation between symbolic entities. In cross-folding, subjects might represent a fold in terms of the regularities presented in Sect. 3. For instance, F_2 can be represented as a nested structure, comprising six schematic 2D forms (cf. Sect. 3.2) which stand in regular spatial relations (cf. Table 1). Additionally, each of those forms can be further decomposed into a spatial configuration of edges. Given such a representation, a successor fold can be realised as a rule-based construction, i. e. a symbol-by-symbol translation according to the between-fold regularities (cf. Table 2).

Crucially, the feasibility of an analytic approach depends on the subject's explicit knowledge of the task domain. Thus, analytic approaches are unlikely to be adopted in the initial phase of the task when subjects are still exploring the ill-defined problem. Only after they have gathered sufficient knowledge, such as the sheet's basic forms and their spatial relations, analytic approaches might start to occur. For example, subjects might utilise their knowledge that some schematic forms are the same for all F_n, i. e. the rectangles, Is and Vs. Beyond that, the transformation of an NV and V into a DV also lends itself to an analytic solution, since it does not involve complex visuospatial manipulations besides aligning two sides of the previous fold (Fig. 10). While analytic approaches thus allow to avoid otherwise complex operations, the lack of visuospatial representations during problem solving can also lead to paradoxical situations: For instance, aligning two wrong sides would yield a symbolic representation of a physically impossible fold state.

As it gets successively more demanding to represent the sides of higher-numbered folds in terms of simple derivations of the basic forms, ultimately an analytic approach requires an even more economical, *syntactic* way of representing sides. This could be achieved by encoding the number of nestings of edges on the open end of a side. For instance, one could use `0` to encode a simple open edge, `1` for a looped edge, `2` for two nested looped edges etc. So `[0,0]` would represent a single V, and `[0,0,0,0,0,0,1,1,4,1]` an NV_5^O (such as on the right of Fig. 11). Based on such a syntax, one could formulate a recursive procedure which – using the sides of F_2 as base cases, and the regularities in Table 2 as transformation rules – can generate the complex sides of any F_n.

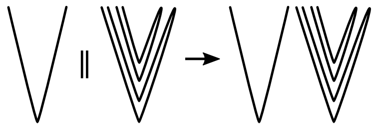

Figure 10: Aligning two existing figures of spatially opposite sides (V_3, NV_3) to form DV_4.

4.3 Mixed Approaches

For the most part however, neither visuospatial nor analytic approaches will likely be used exclusively for any longer period of working on the task. We can rather expect both of them being employed, making use of their respective merits, and with various forms of dynamics taking place between them.

A straight-forward case of interaction would be to generate solutions with one of the approaches, but to employ the other one for checking the results. For instance, since the visuospatial transformation illustrated in Fig. 9 is already quite difficult, subjects might be well advised to check their results explicitly against relevant knowledge. On the other hand, a potentially inadequate posit from a symbolic construction, as in the paradoxical case above, can be verified and corrected with the help of visuospatial manipulations.

But there are more intricate forms of dynamics, as well. Notably, explicit knowledge can be utilised variously in order to *scaffold* or *augment* complexity-bounded visuospatial thinking. For example, instead of one holistic visuospatial representation of the sheet, subjects could use visuospatially less demanding 2D representations of each side, while maintaining the correct spatial relations between them explicitly (cf. Table 1). And even within a visuospatial 2D representation of one of the fold's sides, advanced domain knowledge could also allow decomposing a complex side into simpler sub-figures, which by themselves are easier to deal with visuospatially (folded, rotated etc.) once more (Fig. 11).

Finally, the recognition of perceptually analogous features of different folds (requiring visuospatial representations) might lead to the identification and rule-like representation of general features of cross-folding. For instance, noticing the perceptual similarity between successive NVs (cf. Fig. 5) might lead to the rule-like hypothesis that *all* NVs are composed of two nested sub-figures.

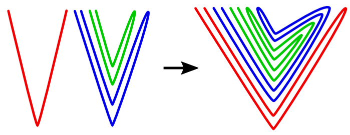

Figure 11: Transforming a DV_4 into an NV_5, colours marking the different sub-figures.

4.4 Summary

Above we have outlined an account of representational dynamics for the proposed task of iterated paper folding. According to this, how subjects approach the task can be expected to vary, dependent on the difficulty of its current stage and the subject's explicit knowledge of the task domain.

While in the initial stages of the ill-defined problem, subjects will likely be successful with visuospatial approaches, in later, more difficult iterations, subjects will profit from changing to more analytic approaches. Therefore, it becomes imperative to gather explicit knowledge of the task domain. However, for the most part subjects will likely follow mixed approaches – as for the relative merits and weaknesses of both visuospatial imagery and knowledge-based constructions. It might thus be more apt to conceive of their development as mutually dependent.

5 Final Remarks

We have presented a new spatial transformation and problem solving task of iterated cross-folding and outlined an analysis of the task domain.

Furthermore, we have discussed how subjects can approach the task domain with a variety of representational forms, each of which can be subject to representational dynamics by themselves, but with dynamics also occurring between these varied approaches [SSB13]. However, the systematic task description provided may have given the impression that subjects actually show equally systematic behaviour when approaching the given task. Yet, according to our prior experience with this task such an assumption would be problematic. For instance, we hardly touched upon the varied roles both perceptual and structural cross-domain analogies can play in scaffolding representations [CPS12, Dun01]. In a similar way, metaphors can also play an important role in changing the task domain's conceptualisation [Ami09]. An overall more naturalistic discussion, describing more ephemeral aspects of working on this task (including "task-extrinsic" aspects such as mind-wandering) is provided in [Sch15].

Regarding the theory of problem solving, the extent of representational dynamics observable in a task such as the one presented here might lead us to doubt the notion of stable problem representations in general. Yet, the question of the potential changeability of problem representations has, for example, also lead to several proposed extensions of problem space theory which try to address the problem by the assumption of additional search processes [KBVK14, KD88, SK95].

Ultimately, in order to make progress in these questions and advance theory, we would need wider-ranging representational-level analyses of subjects solving the presented task, and tasks similar to it. We hope that the task domain and first analyses we presented here have set a good starting point for future endeavours of this kind.

Acknowledgements

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