On separability of the functional space with the open-point and bi-point-open topologies, II

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Abstract

In this paper we continue to study the property of separability of function space $C(X)$ with the open-point and bi-point-open topologies. We show that for every perfect Polish space X a set $C(X)$ with the bi-point-open topology is separable. We also show in the iterated perfect set model that for every regular space X with countable network a set $C(X)$ with bi-point-open topology is separable iff a dispersion character $\Delta(X) = \mathfrak{c}$.

1 Introduction

The space $C(X)$ with the point-open topology (also known as the topology of pointwise convergence) is denoted by $C_p(X)$. It has a subbase consisting of sets of the form $[x,V]^+ = \{f \in C(X) : f(x) \in V\}$, where $x \in X$ and V is an open subset of real line $\mathbb{R}$.

In [3], authors were defined two new kinds of topologies on $C(X)$ now well-known as the open-point and bi-point-open topologies. The open-point topology on $C(X)$ has a subbase consisting of sets of the form

$[U,r]^- = \{f \in C(X) : f^{-1}(r) \bigcap U \neq \emptyset\}$, where U is an open subset of X and $r \in \mathbb{R}$. The open-point topology on $C(X)$ is denoted by h and the space $C(X)$ equipped with the open-point topology h is denoted by $C_h(X)$.

Now the bi-point-open topology on $C(X)$ is the join of the point-open topology p and the open-point topology h. It is the topology having subbasic open sets of both kind: $[x,V]^+$ and $[U,r]^-$, where $x \in X$ and V is an open subset of $\mathbb{R}$, while U is an open subset of X and $r \in \mathbb{R}$. The bi-point-open topology on the space $C(X)$ is denoted by ph and the space $C(X)$ equipped with the bi-point-open topology ph is denoted by $C_{ph}(X)$. One can also view the bi-point-open topology on $C(X)$ as the weak topology on $C(X)$ generated by the identity maps $id_1 : C(X) \mapsto C_p(X)$ and $id_2 : C(X) \mapsto C_h(X)$.

In [3] and [2], the separation and countability properties of these two topologies on $C(X)$ have been studied.

In [3] the following statements were proved.

• $C_h(\mathbb{P})$ is separable where $\mathbb{P}$ is the set of irrational numbers. (Proposition 5.1.)
• If $C_h(X)$ is separable, then every open subset of X is uncountable. (Theorem 5.2.)
• If X has a countable π-base consisting of nontrivial connected sets, then $C_h(X)$ is separable.(Theorem 5.5.)
• If $C_{ph}(X)$ is separable, then every open subset of X is uncountable. (Theorem 5.8.)
• If X has a countable π-base consisting of nontrivial connected sets and a coarser metrizable topology, then $C_{ph}(X)$ is separable. (Theorem 5.10.)

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In [1], there was proved that necessary condition for $C_h(X)$ be a separable space is condition: X has a π-network consisting of $\mathcal{I}$-sets.

- A set $A \subseteq X$ be called *$\mathcal{I}$-set* if there is a continuous function $f \in C(X)$ such that $f(A)$ contains an interval $\mathcal{I} = [a, b] \subset \mathbb{R}$.
- If $C_h(X)$ is a separable space, then X has a π-network consisting of $\mathcal{I}$-sets. (Theorem 2.3.)

In this paper we use the following conventions. The symbols $\mathbb{R}$, $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the space of real numbers, irrational numbers, rational numbers and natural numbers, respectively. Recall that a dispersion character $\Delta(X)$ of X is the minimum of cardinalities of its nonempty open subsets.

Recall also that a space be called Polish space if it is a separable complete metrizable space.

By a *set of reals* we mean a zero-dimensional, separable metrizable space every non-empty open set which has the cardinality the continuum.

2 Main results

Note that if the space $C_h(X)$ is a separable space then $\Delta(X) \geq \mathfrak{c}$. If $A = \{f_i\}$ is a countable dense set of $C_h(X)$ then for each non-empty open set U of X we have $\bigcup f_i(U) = \mathbb{R}$. It follows that $|U| \geq \mathfrak{c}$.

Also note that if the space $C_{ph}(X)$ is a separable space then $C_p(X)$ is a separable space and $C_h(X)$ is separable. It follows that X is a separable submetrizable (coarser separable metric topology) space and $\Delta(X) = \mathfrak{c}$.

Recall that a family γ of subsets of a space X is called T_0-separating if whenever x and y are distinct points of X, there exists $V \in \gamma$ containing exactly one of the points x and y.

In [1], Osipov was proved the following result.

Theorem 2.1. *If X is a Tychonoff space with network consisting non-trivial connected sets, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*
2. *X is a separable submetrizable space.*

In the present paper, we consider more general form this theorem.

Theorem 2.2. *If X is a Tychonoff space with π-network consisting non-trivial connected sets, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*
2. *X has a countable T_0-separating family of zero-sets.*
3. *X is a separable submetrizable space.*

Доказательство. (1) $\Rightarrow$ (2). Let $C_{ph}(X)$ be a separable space. There is a countable dense subset $A = \{f_i\}$ of the space $C_{ph}(X)$. Fix $\beta = \{B_j\}$ some a countable base for $\mathbb{R}$ consisting of bounded open intervals.

Consider $\gamma = \{f^{-1}(\overline{B}) : f \in A, B \in \beta\}$.

We show that the family γ is required family.

Let x_1 and x_2 be distinct points of X. Consider an open base set $Q = [\{x_1\}, (c_1, d_1)]^+ \bigcap [\{x_2\}, (c_2, d_2)]^+$ of the space $C_{ph}(X)$ where $(c_i, d_i) \in \beta$ for $i = 1, 2$ and $\overline{(c_1, d_1)} \bigcap \overline{(c_2, d_2)} = \emptyset$. There is $h \in Q \bigcap A$. Clearly that $h^{-1}(\overline{(c_i, d_i)}) \in \gamma$, $x_i \in h^{-1}(\overline{(c_i, d_i)})$ for $i = 1, 2$ and $h^{-1}(\overline{(c_1, d_1)}) \bigcap h^{-1}(\overline{(c_2, d_2)}) = \emptyset$.

The countable family γ is required family.

(2) $\Rightarrow$ (3). Let $\gamma = \{Z_i\}$ be the countable family with required conditions. We can assume that γ is closed under finite unions. For each $i \in \mathbb{N}$ there is a continuous function $f_i : X \mapsto I = [0, 1]$ such that $Z_i = f_i^{-1}(0)$.

Let $I_i = I \times \{i\}$ for every $i \in \mathbb{N}$. By letting $(x, i_1)E(y, i_2)$ whenever $x = 0 = y$ or $x = y$ and $i_1 = i_2$ we define an equivalence relation E on the set $\bigcup_{i \in \mathbb{N}} I_i$.

The formula

$$\rho([(x, i_1)], [(y, i_2)]) = \begin{cases} |x - y|, \ if \ i_1 = i_2, \\ x + y, \ if \ i_1 \neq i_2, \end{cases}$$

defines a metric on the set of equivalence classes of E. This space - as well as the corresponding metrizable space - be called the *metrizable hedgehog of spininess* $\aleph_0$ and be denoted $J(\omega)$ (Example 4.1.5 in [6]).

Note that for every $i \in \mathbb{N}$ the mapping j_i of the interval I to $J(\omega)$ defined by letting $j_i(x) = [(x,i)]$ is a homeomorphic embedding. The family of all balls with rational radii around points of the form $[(r,i)]$, where r is a rational number, is a base for $J(\omega)$; so that $J(\omega)$ is a separable metrizable space.

The formula $h_i(x) = j_i(f_i(x))$ defines a continuous mapping $h_i : X \mapsto J(\omega)$. Note that the family $\{h_i\}_{i=1}^{\infty}$ is functionally separates points of X. Really let x and y be distinct points of X. There exists $Z \in \gamma$ containing exactly one of the points x and y and there are $i' \in \mathbb{N}$ and continuous function $f_{i'} : X \mapsto I = [0,1]$ such that $Z = f_{i'}^{-1}(0)$. Hence $h_{i'}(x) \neq h_{i'}(y)$. Thus diagonal mapping $h = \triangle_{i\in\mathbb{N}} h_i : X \mapsto J(\omega)^{\omega}$ is a continuous one-to-one mapping from X into the separable metrizable space $J(\omega)^{\omega}$.

It follows that X is a separable submetrizable space.

(3) $\Rightarrow$ (1). Let X be a separable submetrizable space, i.e. X has coarser separable metric topology τ_1 and γ be π-network of X consisting non-trivial connected sets. Let $\beta = \{B_i\}$ be a countable base of (X, τ_1). We can assume that β closed under finite union of its elements.

For each finite family $\{B_{s_i}\}_{i=1}^{d} \subset \beta$ such that $\overline{B_{s_i}} \bigcap \overline{B_{s_j}} = \emptyset$ for $i \neq j$ and $i,j \in \overline{1,d}$ and $\{p_i\}_{i=1}^{d} \subset \mathbb{Q}$ we fix $f = f_{s_1,...,s_d,p_1...,p_d} \in C(X)$ such that $f(\overline{B_{s_i}}) = p_i$ for each $i = \overline{1,d}$.

Let G be the set of functions $f_{s_1,...,s_d,p_1...,p_d}$ where $s_i \in \mathbb{N}$ and $p_i \in \mathbb{Q}$ for $i \in \mathbb{N}$. We claim that the countable set G is a dense set of $C_{ph}(X)$.

By proposition 2.2 in [3], let $W = [x_1, V_1]^+ \bigcap ... \bigcap [x_m, V_m]^+ \bigcap [U_1, r_1]^- \bigcap ... \bigcap [U_n, r_n]^-$ be a base set of $C_{ph}(X)$ where $n, m \in \mathbb{N}$, $x_i \in X$, V_i is open set of $\mathbb{R}$ for $i \in \overline{1,m}$, U_j is open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1,n}$ and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \bigcap \overline{U_j} = \emptyset$.

Choose $B_{s_l} \in \beta$ for $l = \overline{1, m+n}$ such that

1. $\overline{B_{s_{l_1}}} \bigcap \overline{B_{s_{l_2}}} = \emptyset$ for $l_1 \neq l_2$ and $l_1, l_2 \in \overline{1, n+m}$;
2. $x_i \in B_{s_l}$ for $l \in \overline{1,m}$;
3. $B_{s_l} \bigcap U_k \neq \emptyset$ for $l \in \overline{m+1, n+m}$ and $k = l - m$.

Choose $B_{s'_l} \in \beta$ for $l \in \overline{1,m}$ such that $x_i \in B_{s'_l}$ and $\overline{B_{s'_l}} \subseteq B_{s_l}$.

Choose $A_k \in \gamma$ for $k \in \overline{1,n}$ such that $A_k \subseteq (U_k \bigcap B_{s_l})$ where $l = k + m$.

Choose different points $s_k, t_k \in A_k$ for every $k = \overline{1,m}$.

Let $S, T \in \beta$ such that $\overline{S} \bigcap \overline{T} = \emptyset$, $\overline{B_l} \bigcap \overline{S} = \emptyset$, $\overline{B_l} \bigcap \overline{T} = \emptyset$ for $l \in \overline{1,m}$ and $s_k \in S$ and $t_k \in T$ for all $k = \overline{1,m}$.

Fix points $v_i \in (V_i \bigcap \mathbb{Q})$ for $i \in \overline{1,m}$.

Choose $p, q \in \mathbb{Q}$ such that $p < \min\{r_i : i = \overline{1,n}\}$ and $q > \max\{r_i : i = \overline{1,n}\}$.

Let

$$f(x) = \begin{cases} p & for \quad x \in \overline{S} \\ q & for \quad x \in \overline{T} \\ v_l & for \quad x \in \overline{B_{s'_l}} \end{cases}$$

where $l \in \overline{1,m}$.

Note that $f \in W \bigcap G$. This proves theorem.

□

In [1] the following statements were proved.

Theorem 2.3. *(Theorem 2.4 in [1]) Let X be a Tychonoff space with a countable π-base, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*
2. *X is a separable submetrizable space and it has a countable π-network consisting of $\mathcal{I}$-sets.*

Theorem 2.4. *(Corollary 2.5 [1]) Let X be a Tychonoff space with a countable π-base, then the following are equivalent.*

1. *$C_h(X)$ is a separable space.*
2. *X has a countable π-network consisting of $\mathcal{I}$-sets.*

The next result is the corollary of Theorem 2.3, but we notes its as theorem due to the importance in the class of separable metrizable spaces.

Theorem 2.5. *If X is a separable metrizable space, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*

2. *X has a countable π-network consisting of $\mathcal{I}$-sets.*

We have already noted that if a space $C_{ph}(X)$ is a separable space then

- X is a separable submetrizable space;
- X has a π-network consisting of $\mathcal{I}$-sets.

Theorem 2.6. *If X is a separable submetrizable space with a countable π-network consisting of $\mathcal{I}$-sets, then $C_{ph}(X)$ is a separable space.*

Доказательство. The proof analogously to the proof of the implication $((2) \Rightarrow (1))$ in Theorem 2.4 ([1]).

Let $S = \{S_i\}$ be a countable π-network of X consisting of $\mathcal{I}$-sets. By definition of $\mathcal{I}$-sets, for each $S_i \in S$ there is the continuous function $h_i \in C(X)$ such that $h_i(S_i)$ contains an interval $[a_i, b_i]$ of real line. Consider a countable set

$\{h_{i,p,q}(x) = \frac{p-q}{a_i-b_i} * h_i(x) + p - \frac{p-q}{a_i-b_i} * a_i\}$

of continuous functions on X, where $i \in \mathbb{N}$, $p, q \in \mathbb{Q}$. Note that if $h_i(x) = a_i$ then $h_{i,p,q}(x) = p$ and if $h_i(x) = b_i$ then $h_{i,p,q}(x) = q$.

Let $\beta = \{B_j\}$ be a countable base of (X, τ_1) where τ_1 is a separable metraizable topology on X because of X is a separable submetrizable space. For each pair (B_j, B_k) such that $\overline{B_j} \subseteq B_k$ define continuous functions

$$h_{i,p,q,j,k}(x) = \begin{cases} h_{i,p,q}(x) & for \ x \in B_j \\ \mathbf{0} & for \ x \in X \setminus B_k. \end{cases}$$

and for each $v \in \mathbb{Q}$

$$d_{j,k,v}(x) = \begin{cases} v & for \quad x \in B_j \\ \mathbf{0} & for \quad x \in X \setminus B_k. \end{cases}$$

Let G be the set of finite sum of functions $h_{i,p,q,j,k}$ and $d_{j,k,v}$ where $i, j, k \in \mathbb{N}$ and $p, q, v \in \mathbb{Q}$. We claim that the countable set G is a dense set of $C_{ph}(X)$.

By proposition 2.2 in [3], let

$W = [x_1, V_1]^+ \bigcap ... \bigcap [x_m, V_m]^+ \bigcap [U_1, r_1]^- \bigcap ... \bigcap [U_n, r_n]^-$ be a base set of $C_{ph}(X)$ where $n, m \in \mathbb{N}$, $x_i \in X$, V_i is an open set of $\mathbb{R}$ for $i \in \overline{1,m}$, U_j is an open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1,n}$ and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \bigcap \overline{U_j} = \emptyset$.

Fix points $y_j \in U_j$ for $j = \overline{1,n}$ and choose $B_{s_l} \in \beta$ for $l = \overline{1, n+m}$ such that $\overline{B_{s_{l_1}}} \bigcap \overline{B_{s_{l_2}}} = \emptyset$ for $l_1 \neq l_2$ and $l_1, l_2 \in \overline{1, n+m}$ and $x_i \in B_{s_l}$ for $l \in \overline{1,m}$ and $y_j \in B_{s_l}$ for $l \in \overline{m+1, n}$. Choose $B_{s'_l} \in \beta$ for $l \in \overline{1,m}$ such that $x_i \in B_{s'_l}$ and $\overline{B_{s'_l}} \subseteq B_{s_l}$ and choose $B_{s'_l} \in \beta$ for $l \in \overline{m+1, n+m}$ such that $y_j \in \overline{B_{s'_l}} \subseteq B_{s_l}$ where $l = j + m$.

Fix points $v_i \in (V_i \bigcap \mathbb{Q})$ for $i \in \overline{1,m}$ and $p_j, q_j \in \mathbb{Q}$ such that $p_j < r_j < q_j$ for $j = \overline{1,n}$.

Consider $g \in G$ such that

$g = d_{s'_1, s_1, v_1} + ... + d_{s'_m, s_m, v_m} + h_{i_1, p_1, q_1, s'_{m+1}, s_{m+1}} + ... + h_{i_n, p_n, q_n, s'_{m+n}, s_{n+m}}$ where $S_{i_k} \subset B_{s'_l} \bigcap U_k$ for $k = \overline{1,n}$ and $l = k + m$.

Note that $g \in W \bigcap G$. This proves theorem.

□

Corollary 2.7. If X is a perfect Polish space, then $C_{ph}(X)$ is separable.

Доказательство. It follows immediately from fact that any regular closed subset of a space X is a perfect Polish space and it contains some set which is homeomorphic to 2^ω ([7]). It follows that any non-empty open set of X is $\mathcal{I}$-set. □

Note that there is the example such that
- X hasn't countable chain condition, hence, X hasn't countable π-network consisting of $\mathcal{I}$-sets;
- X is a separable submetrizable space;
- $C_{ph}(X)$ is a separable space.

Example 2.8. *(Example 4.3. in [1]) Let* $X = \oplus_{\alpha<\mathfrak{c}}\mathbb{R}_\alpha$ *be a direct sum of real lines* $\mathbb{R}$.

In this connection a natural question arises.

Question 1. Assume that X is a separable submetrizable space with uncountable π-network consisting of $\mathcal{I}$-sets. Does $C_{ph}(X)$ is separable ?

Recall that a set of reals X is *null* if for each positive ϵ there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that $\sum_n diam(I_n) < \epsilon$. A set of reals X has *strong measure zero* if, for each sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive reals, there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that $diam(I_n) < \epsilon_n$ for all n. For example, every Lusin set has strong measure zero.

In [1] (Example 3.1), author was shown that it is consistent with ZFC that exists the separable metrizable space X such that $\Delta(X) = \mathfrak{c}$ and $C_{ph}(X)$ isn't separable.

Example 2.9. (**CH**) *Let X be a set of reals and it has strong measure zero.*

In [8] was shown that it is consistent with ZFC that for any set of reals of cardinality the continuum, there is a (uniformly) continuous map from that set onto the closed unit interval. In fact, this holds in the iterated perfect set model.

In [1] the following statement was proved.

Theorem 2.10. *(the iterated perfect set model)*
If X is a separable metrizable space, then the following are equivalent.

1. *$C_{ph}(X)$ is a separable space.*
2. *$\Delta(X) = \mathfrak{c}$.*

In the present paper, we consider more general form this theorem.

Theorem 2.11. *(the iterated perfect set model) If X is a regular space with a countable network, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*
2. *$\Delta(X) = \mathfrak{c}$.*
3. *X has a countable π-network consisting of $\mathcal{I}$-sets.*

Доказательство. (1) $\Rightarrow$ (2). Note that if the space $C_{ph}(X)$ is a separable space then $C_p(X)$ is a separable space and $C_h(X)$ is separable. It follows that X is a separable submetrizable space and, hence, $\Delta(X) = \mathfrak{c}$.

(2) $\Rightarrow$ (3). Let $\Delta(X) = \mathfrak{c}$.

(I). We show that any separable metrizable space M of cardinality $\mathfrak{c}$ is $\mathcal{I}$-set of M, i.e. there exists a continuous function $f : M \mapsto \mathbb{R}$ such that $f(M) \supseteq \mathcal{I}$.

Really, if a real-valued continuous image of space M has cardinality less $\mathfrak{c}$ for any $f \in C(M)$, then M is a zero-dimensional space. It follows that M is a set of reals and, by the iterated perfect set model, there is a continuous map from this set onto the closed unit interval $\mathcal{I}$.

If there is a real-valued continuous image of space M such that it has cardinality $\mathfrak{c}$, then either it contains an interval $\mathcal{I}$ or it is a set of reals and, again, by the iterated perfect set model, there is a continuous map from this set onto the closed unit interval $\mathcal{I}$.

(II). Recall that a regular space with a countable network is normal and a separable submetrizable space. Since $\Delta(X) = \mathfrak{c}$ and X is a regular space with a countable network, it follows that X has countable π-network α consisting of closed sets of cardinality $\mathfrak{c}$. We show that α is required π-network.

Let f be a condensation from X onto a separable metrizable space. Fix $A \in \alpha$ and consider the mapping $h = f \restriction A$. By point (I), $h(A)$ is $\mathcal{I}$-set of $h(A)$, i.e. there exist the continuous function $f : h(A) \mapsto \mathbb{R}$ such that $f(h(A)) \supseteq \mathcal{I}$. Since X is a normal space, by Tietze-Urysohn Extension Theorem, the mapping $f \circ h$ can be extended to a real-valued continuous map $F : X \mapsto \mathbb{R}$. Note that $F(A) = f(h(A)) \supseteq \mathcal{I}$ i.e. A is $\mathcal{I}$-set of X.

(2) $\Rightarrow$ (3). It follows from Theorem 2.6.

□

Remark 2.12. The main results of this paper were announced in:

https://arxiv.org/abs/1604.04609. Since then, several remarkable articles ([4],[5]) on the separability of a function space $C(X)$ with the open-point, bi-point-open, bi-compact-open topologies have been published.

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