

# Compact difference scheme for semi-linear delayed diffusion wave equation with fractional order in time

Ahmed S. Hendy<sup>1,2</sup>  
ahmed.hendy@fsc.bu.edu.eg

Vladimir G. Pimenov<sup>1,3</sup>  
v.g.pimenov@urfu.ru

1 – Institute of Natural Sciences and Mathematics, Ural Federal University (Yekaterinburg, Russia)

2 – Department of Mathematics, Benha University (Benha, Egypt)

3 – Krasovskii Institute of Mathematics and Mechanics (Yekaterinburg, Russia)

## Abstract

A compact difference scheme for a class of non-linear fractional diffusion-wave equations with fixed time delay is considered. Analysis of the constructed difference scheme is done in  $L_\infty$ -norm by means of the discrete energy method. A numerical test example is introduced to illustrate the accuracy and efficiency of the proposed method.

## 1 Introduction

An important class of fractional differential equations which has been studied widely in recent years is the time fractional diffusion-wave equation (FDWE). The time FDWE is obtained from the classical diffusion-wave equation by replacing the second-order time derivative term by a fractional derivative of order  $1 < \alpha \leq 2$ . The fractional diffusion equation was introduced in physics by Nigmatullin [16] to describe diffusion in media with fractal geometry, which is a special type of porous media. Gorenflo et al. [10] presented the scale-invariant solutions for the time-fractional diffusion-wave equation in terms of the generalized Wright function. Agrawal [2] extended this formulation to a diffusion-wave equation that contains a fourth-order space derivative term in a bounded space domain. Recently, simulations of the approximation solutions of time-fractional wave, forced wave (shear wave) and damped wave equations are given in [1]. A novel fractional diffusion-wave equation with non-local regularization for noise removal was presented in [24]. The existence and uniqueness of solutions for Dirichlet initial-boundary value problem associated to the semi linear fractional wave equation was recently studied in [14]. As a numerical approach to solve FDWE, Sun and his co-authors proposed a high order difference methods for the fractional diffusion-wave equation [8]. Also, in [22], the efforts of the authors were devoted to the application of fractional multi-step method to obtain a numerical solution of time fractional diffusion-wave equation. Compact finite difference schemes for the modified anomalous fractional sub-diffusion equation and fractional diffusion-wave equation were studied in [21]. A numerical solution for a general class of diffusion problem was considered in [3], where the standard time derivative is replaced by a fractional one. An efficient numerical method was constructed in [26] to solve this moving boundary problem.

Time delay occurs in many realistic applications which are modeled mathematically, e.g. [9, 6, 13, 4, 15, 5]. Reaction-diffusion equations with time delay effect have been proposed as models for the population ecology, the cell biology and the control theory in recent years [19]. The existence of mild solutions for initial value problem for nonlinear time fractional non-autonomous evolution equations with delay in Banach space  $E$  was studied in [7]. Numerically, a linearized quasi-compact difference scheme was proposed for semi-linear space-fractional

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diffusion equations with a fixed time delay [11]. A linearized compact finite difference scheme was presented for the semi-linear fractional delay convection-reaction–diffusion equation in [23]. The authors of the manuscript at hand, recently proposed a difference scheme for a class of non-linear delay distributed order fractional diffusion equations in [17]. As an extension to this contribution and depending on Sun’s work [20], we seek to derive a compact linear difference scheme to solve numerically FDWE effected with a non-linear delayed source function, more specific we consider

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = K \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t, u(x, t), u(x, t - s)), \quad t > 0, \quad 0 \leq x \leq L, \quad (1a)$$

with the following initial and boundary conditions

$$u(x, t) = \tilde{r}(x, t), \quad 0 \leq x \leq L, \quad t \in [-s, 0], \quad \frac{\partial u(x, 0)}{\partial t} = \tilde{\psi}(x) = \lim_{t \rightarrow -0} \frac{\partial \tilde{r}(x, t)}{\partial t}, \quad (1b)$$

$$u(0, t) = \phi_0(t), \quad u(L, t) = \phi_L(t), \quad t > 0, \quad (1c)$$

where  $s > 0$  is the delay parameter,  $K$  is a positive constant. The fractional derivative of order  $1 < \alpha \leq 2$  is defined in Caputo sense.

In order to transform (1) to a system with zero Dirichlet boundary conditions, we define  $h(x, t) := \phi_0(t) + \frac{x}{L}(\phi_L(t) - \phi_0(t))$  and introduce the new function  $v(x, t) = u(x, t) - h(x, t)$ . Hence, we have

$$\frac{\partial^\alpha v(x, t)}{\partial t^\alpha} = K \frac{\partial^2 v(x, t)}{\partial x^2} + f(x, t, v(x, t), v(x, t - s)), \quad t > 0, \quad 0 \leq x \leq L, \quad (2a)$$

with the following initial and boundary conditions

$$v(x, t) = r(x, t), \quad 0 \leq x \leq L, \quad t \in [-s, 0], \quad \frac{\partial v(x, 0)}{\partial t} = \psi(x) = \lim_{t \rightarrow -0} \frac{\partial r(x, t)}{\partial t}, \quad (2b)$$

$$v(0, t) = v(L, t) = 0, \quad t > 0. \quad (2c)$$

We need to overcome two degrees of complexity; how to employ a suitable approximation of the time fractional derivative on the one hand, and how to approximate the non linear delay source function linearly on the other, in order to obtain a numerical solution for (2). Throughout this work and by the aid of [23], we suppose that the function  $f(x, t, \mu, v)$  and the solution  $u(x, t)$  of (2) are sufficiently smooth in the following sense:

- Let  $m$  be an integer satisfying  $ms \leq T < (m + 1)s$ , define  $I_r = (rs, (r + 1)s)$ , for  $r = -1, 0, \dots, m - 1$ ,  $I_m = (ms, T)$ ,  $I = \bigcup_{q=-1}^m I_q$  and assume that  $u(x, t) \in C^{(6,3)}([0, L] \times [0, T])$ ,
- The partial derivatives  $f_\mu(x, t, \mu, v)$  and  $f_v(x, t, \mu, v)$  are continuous in the  $\epsilon_0$ -neighborhood of the solution. Define

$$c_1 = \sup_{\substack{0 < x < L, 0 < t \leq T \\ |\epsilon_1| \leq \epsilon_0, |\epsilon_2| \leq \epsilon_0}} |f_\mu(x, t, u(x, t) + \epsilon_1, u(x, t - s) + \epsilon_2)|, \quad (3a)$$

$$c_2 = \max_{\substack{0 < x < L, 0 < t \leq T \\ |\epsilon_1| \leq \epsilon_0, |\epsilon_2| \leq \epsilon_0}} |f_v(x, t, u(x, t) + \epsilon_1, u(x, t - s) + \epsilon_2)|. \quad (3b)$$

The structure of this paper is arranged as: a derivation of the linear difference scheme is done in the following section. Next, in the third section, the solvability, convergence and stability for the difference scheme are carried out. In the fourth section, numerical examples are given to illustrate the accuracy of the presented scheme and to support our theoretical results.

## 2 Construction of the difference scheme

A numerical solution based on the Crank-Nicholson method is derived. Before we continue, some further notations are fixed. Take two positive integers  $M$  and  $n$ , let  $h = \frac{L}{M}$ ,  $\tau = \frac{s}{n}$  and denote  $x_i = i h$  for  $i = 0, \dots, M$ ;  $t_k = k \tau$  and  $t_{k-1/2} = (k - \frac{1}{2}) \tau = \frac{1}{2}(t_k + t_{k-1})$ , for  $k = -n, \dots, N$ , where  $N = \lfloor \frac{T}{\tau} \rfloor$ . Using the points  $x_i$  in space and  $t_k$  in time we cover the space-time domain by  $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$ , where  $\Omega_h = \{x_i \mid 0 \leq i \leq M\}$  and  $\Omega_\tau = \{t_k \mid -n \leq k \leq N\}$ .

Let  $\mathcal{W} = \{w : \Omega_{h\tau} \rightarrow \mathbb{R} \mid w(x_i, t_k) = W_i^k; i = 0, 1, \dots, M; k = -n, -n+1, \dots, N\}$  be a grid function space on  $\Omega_{h\tau}$ . For  $w \in \mathcal{W}$ , we define  $W_i^{k-1/2} := \frac{1}{2}(W_i^k + W_i^{k-1})$ .

**Lemma 1.** Let  $q(x) \in C^6([x_{i-1}, x_{i+1}])$ , then

$$\frac{1}{12} (q''(x_{i-1}) + 10q''(x_i) + q''(x_{i+1})) - \frac{1}{h^2} (q(x_{i-1}) - 2q(x_i) + q(x_{i+1})) = \frac{h^4}{240} q^{(6)}(\omega_i),$$

where  $\omega_i \in (x_{i-1}, x_{i+1})$ . [25]

In [20], an approximation for the time Caputo fractional derivative at  $t_{k-1/2}$  with  $1 < \alpha < 2$  was given:

$$\frac{\partial^\alpha v(x_i, t_{k-1/2})}{\partial t^\alpha} = \frac{1}{\bar{\mu}} \left( b_0^\alpha \delta_t V_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t V_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) + r_k, \quad (4a)$$

where  $\psi(x)$  is defined in (2b),

$$b_k^\alpha = \frac{\tau^{2-\alpha}}{2-\alpha} ((k+1)^{2-\alpha} - k^{2-\alpha}), \quad \bar{\mu} = \tau \Gamma(2-\alpha), \quad (4b)$$

$$|r_k| \leq \frac{1}{\Gamma(3-\alpha)} \left( \frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha}) + \frac{1}{12} \right) \max_{0 \leq t \leq t_k} \left| \frac{\partial^3 v}{\partial t^3} \right| \tau^{3-\alpha}, \quad (4c)$$

and for any function  $v : [0, L] \times [-s, +\infty) \rightarrow \mathbb{R}$  one denotes  $v(x_i, t_j) = V_i^j$  for  $i \in \mathbb{N}$ ,  $j \in \mathbb{Z}$  and defines

$$V_i^{k-1/2} = \frac{1}{2} (V_i^k + V_i^{k-1}), \quad \delta_t V_i^{k-1/2} = \frac{1}{\tau} (V_i^k - V_i^{k-1}), \quad (4d)$$

$$\delta_x V_{i-1/2}^k = \frac{1}{h} (V_i^k - V_{i-1}^k), \quad \delta_x^2 V_i^k = \frac{1}{h^2} (V_{i+1}^k - 2V_i^k + V_{i-1}^k). \quad (4e)$$

We are now in a position to apply and combine the above, that is (4), to (2a) at the points  $(x_i, t_{k-1/2})$ , and arrive at

$$\left[ \frac{1}{\bar{\mu}} \left( b_0^\alpha \delta_t V_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t V_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) + r_k \right] = K \frac{\partial^2 v(x_i, t_{k-1/2})}{\partial x^2} + f(x_i, t_{k-1/2}, v(x_i, t_{k-1/2}), v(x_i, t_{k-1/2} - s)), \quad (5)$$

such that  $i = 0, \dots, M$ ,  $k = 1, \dots, N$ .

**Lemma 2.** For  $g = (g_0, g_1, \dots, g_M)$ , let the linear operator  $\mathfrak{A}$  be defined as

$$\mathfrak{A}g_i = \frac{1}{12} (g_{i-1} + 10g_i + g_{i+1}), \quad 1 \leq i \leq M-1.$$

Then, we obtain

$$\mathfrak{A} \left[ \frac{1}{\bar{\mu}} \left( b_0^\alpha \delta_t V_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t V_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) \right] = K \delta_x^2 V_i^{k-1/2} + \mathfrak{A}f \left( x_i, t_{k-1/2}, \frac{3}{2} V_i^{k-1} + \frac{1}{2} V_i^{k-2}, \frac{1}{2} V_i^{k-n-1} + \frac{1}{2} V_i^{k-n} \right) + R_i^{k-1/2}, \quad (6)$$

where

$$\left| R_i^{k-1/2} \right| \leq \bar{C} (\tau^{3-\alpha} + h^4), \quad 1 \leq i \leq M-1, \quad 1 \leq k \leq N. \quad (7)$$

*Proof.* By using the following Taylor expansions

$$\begin{aligned}\frac{\partial^2 v(x_i, t_{k-1/2})}{\partial x^2} &= \frac{1}{2} \left( \frac{\partial^2 v(x_i, t_k)}{\partial x^2} + \frac{\partial^2 v(x_i, t_{k-1})}{\partial x^2} \right) + O(\tau^2), \\ v(x_i, t_{k-1/2}) &= \frac{3}{2} V_i^{k-1} - \frac{1}{2} V_i^{k-2} + O(\tau^2), \\ v(x_i, t_{k-1/2} - s) &= \frac{1}{2} V_i^{k-n} + \frac{1}{2} V_i^{k-n-1} + O(\tau^2),\end{aligned}$$

in (5) we obtain

$$\left[ \frac{1}{\mu} \left( b_0^\alpha \delta_t V_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t V_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) + r_k \right] = \frac{K}{2} \left( \frac{\partial^2 v(x_i, t_k)}{\partial x^2} + \frac{\partial^2 v(x_i, t_{k-1})}{\partial x^2} \right) + f \left( x_i, t_{k-1/2}, \frac{3}{2} V_i^{k-1} - \frac{1}{2} V_i^{k-2}, \frac{1}{2} V_i^{k-n} + \frac{1}{2} V_i^{k-n-1} \right) + O(\tau^2), \quad (8)$$

where we used the continuity of the derivatives of  $f$  in its third and fourth component when letting  $\tau \rightarrow 0$ . According to Lemma 1, we have

$$\mathfrak{A} \left[ \frac{\partial^2 v(x_i, t_k)}{\partial x^2} \right] = \delta_x^2 V_i^k + \frac{h^4}{240} \frac{\partial^6 v}{\partial x^6}(\theta_i^k, t_k), \quad \theta_i^k \in (x_{i-1}, x_{i+1}),$$

so applying  $\mathfrak{A}$  to (8) we arrive at, using (4e),

$$\begin{aligned}\mathfrak{A} \left[ \frac{1}{\mu} \left( b_0^\alpha \delta_t V_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t V_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) + r_k \right] &+ O(h^4) + O(\tau^2) \\ &= \frac{K}{2} \delta_x^2 (V_i^k + V_i^{k-1}) + \mathfrak{A} f \left( x_i, t_{k-1/2}, \frac{3}{2} V_i^{k-1} - \frac{1}{2} V_i^{k-2}, \frac{1}{2} V_i^{k-n} + \frac{1}{2} V_i^{k-n-1} \right),\end{aligned}$$

as  $v(x, t) \in C^{(6,3)}([0, L] \times [0, T])$ . Define  $R_i^{k-1/2} = \mathfrak{A} r_k + O(h^4) + O(\tau^2)$ , then from (4c), the estimate (7) is achieved and the proof is complete.  $\blacksquare$

The final form of our difference scheme is obtained by neglecting  $R_i^{k-1/2}$  and replacing  $V_i^k$  with  $v_i^k$  in (6)

$$\begin{aligned}\mathfrak{A} \left[ \frac{1}{\mu} \left( b_0^\alpha \delta_t v_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t v_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) \right] \\ = K \delta_x^2 v_i^{k-1/2} + \mathfrak{A} f \left( x_i, t_{k-1/2}, \frac{3}{2} v_i^{k-1} - \frac{1}{2} v_i^{k-2}, \frac{1}{2} v_i^{k-n-1} + \frac{1}{2} v_i^{k-n} \right), \quad (9a)\end{aligned}$$

such that  $1 \leq i \leq M-1$ ,  $1 \leq k \leq N$ , and supplying appropriate initial and boundary conditions

$$v_0^k = \phi_0(t_k), \quad v_M^k = \phi_L(t_k), \quad 1 \leq k \leq N, \quad (9b)$$

$$v_i^k = r(x_i, t_k), \quad 0 \leq i \leq M, \quad -n \leq k \leq 0. \quad (9c)$$

Recall that  $v(x_i, t_k) = V_i^k$  and  $v_i^k$  is the solution of the difference scheme, hoping to have  $v(x_i, t_k) \approx v_i^k$ , we discuss in the next section  $\epsilon_i^k := |V_i^k - v_i^k|$ .

We now prove that our difference scheme admits a unique solution. Next, we show that the obtained solution solves (2).

**Theorem 1. (Solvability).** *The difference scheme (9) is uniquely solvable.*

*Proof.* We can arrange the system (9) as follows

$$\begin{aligned}
& \left[ \frac{1}{12} \frac{b_0^\alpha}{\bar{\mu}\tau} - \frac{K}{2h^2} \right] v_{i+1}^k + \left[ \frac{10}{12} \frac{b_0^\alpha}{\bar{\mu}\tau} + \frac{K}{h^2} \right] v_i^k + \left[ \frac{1}{12} \frac{b_0^\alpha}{\bar{\mu}\tau} - \frac{K}{2h^2} \right] v_{i-1}^k \\
&= \left[ \frac{1}{12} \bar{\mu}\tau(2b_0^\alpha - b_1^\alpha) - \frac{K}{2h^2} \right] v_{i+1}^{k-1} + \left[ \frac{10}{12} \bar{\mu}\tau(2b_0^\alpha - b_1^\alpha) + \frac{K}{h^2} \right] v_i^{k-1} \\
&\quad + \left[ \frac{1}{12} \bar{\mu}\tau(2b_0^\alpha - b_1^\alpha) - \frac{K}{2h^2} \right] v_{i-1}^{k-1} \\
&+ \bar{\mu}\tau \mathfrak{A} \left( \sum_{j=1}^{k-2} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) v_i^j - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) v_i^{j-1} + \tau b_{k-1}^\alpha \psi_i \right) \\
&\quad + \mathfrak{A} f \left( x_i, t_{k+1/2}, \frac{3}{2} v_i^{k-1} - \frac{1}{2} v_i^{k-2}, \frac{1}{2} v_i^{k+1-n} + \frac{1}{2} v_i^{k-n} \right), \quad (10)
\end{aligned}$$

or written in a more concise form

$$\mathbf{A}v^k = \varphi_k(v^{k-1}, v^{k-1}, \dots, v^{-n}).$$

The tridiagonal coefficient matrix  $\mathbf{A} = (a_{ij})$  is strictly diagonally dominant because  $a_{ii} > \sum_{j \neq i} |a_{ij}|$ ;

$$a_{ii} = \frac{10}{12} \frac{b_0^\alpha}{\bar{\mu}\tau} + \frac{K}{h^2}, \quad a_{i+1,i} = \frac{1}{12} \frac{b_0^\alpha}{\bar{\mu}\tau} - \frac{K}{2h^2} = a_{i-1,i}.$$

Therefore, the coefficient matrix is nonsingular and the theorem is readily proved by strong induction.  $\blacksquare$

### 3 Convergence and stability for the difference scheme

Now, we introduce the uniqueness, stability and convergence theorems in  $L_\infty$ -norm using the discrete energy method for the proposed difference scheme.

The spatial domain  $[0, L]$  is covered by  $\Omega_h = \{x_i \mid 0 \leq i \leq M, \}$  and let  $V_h = \{v \mid v = (v_0, \dots, v_M), \quad v_0 = v_M = 0\}$  be a grid function space on  $\Omega_h$ . For any  $u, v \in V_h$ , define the discrete inner products and corresponding norms as

$$\begin{aligned}
\langle u, v \rangle &= h \sum_{i=1}^{M-1} u_i v_i, \quad \langle \delta_x u, \delta_x v \rangle = h \sum_{i=1}^M (\delta_x u_{i-1/2}) (\delta_x v_{i-1/2}), \\
\langle \delta_x^2 u, v \rangle &= -\langle \delta_x u, \delta_x v \rangle, \quad \delta_x u_i = \frac{1}{h} (u_i - u_{i-1}), \\
\|u\| &= \sqrt{\langle u, u \rangle}, \quad |u|_1 = \sqrt{\langle \delta_x u, \delta_x u \rangle}, \quad \|u\|_\infty = \max_{0 \leq i \leq M} |u|,
\end{aligned}$$

and denote

$$|v|_1 = \sqrt{h \sum_{i=1}^M (\delta_x v_i)^2}, \quad \|\delta_x^2 v\|_\infty = \sqrt{h \sum_{i=1}^{M-1} (\delta_x^2 v_i)^2}.$$

According to [25], the following inequalities are fulfilled

$$\|u\|_\infty \leq \frac{\sqrt{L}}{2} |u|_1, \quad \|u\| \leq \frac{L}{\sqrt{6}} |u|_1. \quad (11)$$

For the analysis of the difference scheme, we will use the following lemmas:

**Lemma 3.** *Let  $v \in V_h$ , we have  $\|\delta_x^2 v\| \leq \frac{2}{h} |v|_1$ .*

**Lemma 4.** [18] *Let  $v \in V_h$  and  $v_0 = v_M = 0$ . Then, we have  $\|v\|_\infty \leq \frac{\sqrt{L}}{2} |v|_1$ .*

**Lemma 5.** [20] *For any  $G = \{G_1, G_2, G_3, \dots\}$  and  $\psi$ , we obtain*

$$\sum_{k=1}^m \left[ b_0^\alpha G_k - \sum_{j=1}^{k-1} (b_{k-j-1}^{\alpha_l} - b_{k-j}^{\alpha_l}) G_j - b_{k-1}^{\alpha_l} \psi \right] G_k \geq \frac{t_m^{1-\alpha_l}}{2(2-\alpha_l)} \tau \sum_{k=1}^m G_k^2 - \frac{t_m^{2-\alpha_l}}{2(2-\alpha_l)} \psi^2.$$

**Lemma 6.** Gronwall inequality [25]. Suppose that  $\{F^k \mid k \geq 0\}$  is a nonnegative sequence and satisfies  $F^{k+1} \leq A + B\tau \sum_{l=1}^k F^l$ ,  $k \geq 0$ , for some nonnegative constants  $A$  and  $B$ . Then  $F^{k+1} \leq A \exp(Bk\tau)$ .

**Theorem 2. (Convergence).** Let  $v(x, t) \in [0, L] \times [-s, T]$ , be the solution of (2) such that  $v(x_i, t_k) = V_i^k$  and  $v_i^k$  ( $0 \leq i \leq M, -n \leq k \leq N$ ) be the solution of the difference scheme (9), denote  $e_i^k = V_i^k - v_i^k$ , for  $0 \leq i \leq M, -n \leq k \leq N$ ,

$$C = \sqrt{\frac{3L}{8\theta K}} \exp\left(\frac{3T}{8\theta K}(5c_1^2 + c_2^2)\right), \quad \theta = \frac{1}{2}T^{\alpha-1}\Gamma(3-\alpha),$$

then if

$$\tau \leq \tau_0 = \left(\frac{\epsilon_0}{4C}\right)^{\frac{1}{3-\alpha}}, \quad h \leq h_0 = \left(\frac{\epsilon_0}{4C}\right)^{\frac{1}{4}}, \quad (12)$$

one has that

$$\|e^k\|_{\infty} \leq C(\tau^{3-\alpha} + h^4), \quad 0 \leq k \leq N, \quad (13)$$

where  $c_1, c_2$  and  $\epsilon_0$  are those from (3).

The proof uses the previous formulated lemmas in the sense of our results in [12].

To discuss the stability of the difference scheme (9), we also use the discrete energy method in the same way like the discussion of the convergence. Let  $\{z_i^k \mid 0 \leq i \leq M, 0 \leq k \leq N\}$  be the solution of

$$\begin{aligned} \mathfrak{A} \left[ \frac{1}{\mu} \left( b_0^\alpha \delta_t z_i^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-j-1}^\alpha - b_{k-j}^\alpha) \delta_t z_i^{j-1/2} - b_{k-1}^\alpha \psi(x_i) \right) \right] \\ = K \delta_x^2 z_i^{k-1/2} + \mathfrak{A}f \left( x_i, t_{k-1/2}, \frac{1}{2}z_i^k + \frac{1}{2}z_i^{k-1}, \frac{1}{2}z_i^{k-n-1} + \frac{1}{2}z_i^{k-n} \right), \end{aligned} \quad (14a)$$

such that  $1 \leq i \leq M-1, 1 \leq k \leq N$ , and supplied by appropriate initial and boundary conditions

$$z_0^k = \phi_0(t_k), \quad z_M^k = \phi_L(t_k), \quad 1 \leq k \leq N, \quad (14b)$$

$$z_i^k = r(x_i, t_k) + \rho_i^k, \quad 0 \leq i \leq M, \quad -n \leq k \leq 0, \quad (14c)$$

where  $\rho_i^k$  is the perturbation of  $\psi(x_i, t_k)$ .

Following the same steps as in the proof of the convergence theorem, the stability of the scheme is obtained.

**Theorem 3. (Stability).** Let  $\theta_i^k = z_i^k - v_i^k$ , for  $0 \leq i \leq M, -n \leq k \leq N$ . Then there exist some arbitrary positive constants  $c_4, c_5, h_0, \tau_0$  which fulfill

$$\|\theta^k\|_{\infty} \leq c_4 \sqrt{\tau} \sum_{k=-n}^0 \|\rho^k\|, \quad 0 \leq k \leq N, \quad \|\rho^k\| = \sqrt{h \sum_{i=1}^{M-1} (\rho_i^k)^2},$$

conditioned by

$$h \leq h_0, \quad \tau \leq \tau_0, \quad \max_{\substack{-n \leq k \leq 0 \\ 0 \leq i \leq M}} |\rho_i^k| \leq c_5.$$

## 4 Numerical Verification

Let  $v_i^k$  be the solution of the constructed difference scheme (9) with the step sizes  $\tau$  and  $h$ . Define the maximum norm error by  $E(\tau, h) = \max_{\substack{0 \leq i \leq M \\ 0 \leq k \leq N}} \|V_i^k - v_i^k\|_{\infty}$ . Also, define the following error rates

$$rate_1 = \log_2 \left( \frac{E(2\tau, h)}{E(\tau, h)} \right), \quad rate_2 = \log_2 \left( \frac{E(\tau, 2h)}{E(\tau, h)} \right).$$

Consider the following numerical test example

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t, u(x, t), u(x, t-s)), \quad t \in (0, 1), \quad 0 < x < \pi, \quad (15a)$$

$f(x, t, u(x, t), u(x, t - s)) = \sin(x) \left( (t^3 + 2t + 4) + \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} \right) - u(x, t - s) + \sin(x) ((t - s)^3 + 2(t - s) + 4)$ ,  
with the following initial and boundary conditions

$$u(x, t) = (t^3 + 2t + 4) \sin(x), \quad 0 \leq x \leq \pi, \quad t \in [-s, 0), \quad s > 0, \quad (15b)$$

$$u(0, t) = u(\pi, t) = 0, \quad t \in [0, 1]. \quad (15c)$$

The exact solution of this problem is

$$u(x, t) = (t^3 + 2t + 4) \sin(x). \quad (16)$$

Results are presented in Tables (1) - (2) (time) and Table 3 (space). From these numerical results, we can see a good agreement between theoretical and numerical results.

Table 1: Errors and convergence order of the difference scheme (9) for (15) in time variable with  $\alpha = 1.25$ ,  $h = \pi/3000$  and with time delay  $s = 1$ .

$\tau$	$E(\tau, h)$	$rate_1$
$\frac{1}{10}$	0.00015	
$\frac{1}{20}$	0.00004	1.732
$\frac{1}{40}$	0.00001	1.738
$\frac{1}{80}$	$4.049 \times 10^{-6}$	1.741
$\frac{1}{160}$	$1.209 \times 10^{-6}$	1.744
$\frac{1}{320}$	$3.597 \times 10^{-7}$	1.749

Table 2: Errors and convergence order of the difference scheme (9) for (15) in time variable with  $\alpha = 1.75$ ,  $h = \pi/3000$  and with time delay  $s = \frac{1}{4}$ .

$\tau$	$E(\tau, h)$	$rate_1$
$\frac{1}{40}$	0.00003	
$\frac{1}{80}$	0.00001	1.232
$\frac{1}{160}$	$5.415 \times 10^{-6}$	1.238
$\frac{1}{320}$	$2.288 \times 10^{-6}$	1.243
$\frac{1}{640}$	$9.625 \times 10^{-7}$	1.249

Table 3: Errors and convergence order of the difference scheme (9) for (15) in space variable with  $\alpha = 1.5$ ,  $\tau = 1/10000$  and with time delay  $s = \frac{1}{2}$ .

$h$	$E(\tau, h)$	$rate_2$
$\frac{\pi}{4}$	0.00027	
$\frac{\pi}{8}$	0.000018	3.872
$\frac{\pi}{16}$	$1.2525 \times 10^{-6}$	3.880
$\frac{\pi}{32}$	$8.24587 \times 10^{-8}$	3.925
$\frac{\pi}{64}$	$5.28023 \times 10^{-9}$	3.965
$\frac{\pi}{128}$	$3.33926 \times 10^{-10}$	3.983

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