An Algorithm for Exact Geometric Search of Polynomials Complex Roots

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Abstract. A graph of an n-th order polynomial on the real plane allows us to define geometrically all the real roots. Number of real roots varies from 0 to n. The rest of the roots are complex and not determined by the graph. In the article, in addition to the graph of the basic polynomial, two auxiliary graphs are constructed, which allow us to represent all complex roots on the same real plane. The application of this method is considered in detail for the solution of a cubic polynomial. In this case the method has exceptional features in comparison with polynomials of other degrees. Taking into account the well-known expressions for the roots of polynomials of order 3 and 4, the auxiliary graphs of the method have exact formulas for polynomials with order $n \leq 10$.

Keywords: polynom, complex roots, function graph , exact algorythm

1 Introduction

Complex numbers were introduced as an auxiliary tool for finding the real roots of polynomials 3 rd and 4 th orders. The basic algebraic theorem proves that the number of complex and real roots is equal to order of the polynomial. The real roots are easy to find and illustrated on the graph. Complex roots do not possess this property. There are several exact algebraic and iterative numerical methods for finding the roots ([1] —[4]). Precise methods for polynomials with arbitrary coefficients, as is known, exist only for polynomials of order $n \leq 4$. These are the well-known Cardano ($n = 3$) and Torriceli ($n = 4$) formulas.

In this article, the exact geometric method is understood as the use of graphs of functions that have an exact analytical expression, and a one-sided ruler with the ability to draw parallel lines. Along with the graph of the original polynomial $f(x)$, we construct a graph of auxiliary multivalued "conjugate" function $f_S(x)$. On this graph, a subset is distinguished, which is called the gluing set and is denoted S_f. Then the second auxiliary multivalued "carrier" function $f_N(x)$ is constructed. The domain of definition of $f_N(x)$ coincides with the projection of the gluing set S_f onto the axis Ox. To find the complex roots of the equation $f(x) = \mathit{\Delta}$, where $\mathit{\Delta}$ is an arbitrary real number, we use the same real plane. The Ox axis is associated with the real part of the complex number, and the Oy axis is joined with the imaginary part. Both auxiliary functions $f_S(x)$ and $f_N(x)$ have an exact analytic notation for $n \leq 10$. For polynomials of higher degrees, the graphs of the auxiliary functions must be constructed numerically. The algorithm assumes drawing the straight lines. First, we draw a horizontal line with the equation $y = \varDelta$ and find the intersection points with the glueing set. Then through each such point, we draw a vertical line and find the points of intersection with the carrier graph. These last points are complex roots of the equation $f(x) = \varDelta$. When the free coefficient $\varDelta$ changes, the method shows the migration way [8], in which the roots move. We can see how a real root splits into two complex ones, and vice versa, two complex conjugate roots join into one real root. The geometric method for solving the cubic equation is studied in detail in the article. In this case, the method has exceptional features in comparison with polynomials of other degrees, namely, the auxiliary functions are expressed in terms of the original polynomial $f(x)$. It is shown that the conjugate function $f_S(x) = f(-2x - a_1/a_0)$, and the carrier function $f_N(x)$ is expressed in terms of the derivative of the original polynomial $f_N(x) = \sqrt{f'(x)}$. Moreover, the conjugate function is a single-valued function, while for other degrees it is multivalued.

The developed method can be used, for example, for an exact geometric representation of the root locus in the study of control systems with feedback, in which there is a parametric block multiplier k. The parameter k plays the role of the free coefficient $\varDelta$ and the problem is to find all the poles of the characteristic polynomial of a closed system ([5] —[8]).

2 Basic concepts

Definition 1. *Let $f(x)$ be a polynomial of order n with real coefficients. The gluing set of $f(x)$ is called the set S_f of points (x, y) of the real plane $\mathbb{R}^2$, for which there are distinct complex mutually conjugate numbers z_1 and z_2 such that $Re\ z_1 = Re\ z_2 = x$ and $f(z_1) = f(z_2) = y$.*

The gluing set is the set of intersection points (or gluing) of the four-dimensional graph of the complex function $f(z)$ of the complex variable z ([9]).

Lemma 1. *Let $f(x)$ be a polynomial of the third order. If the point $(a, \varDelta) \in S_f$, then the equation*

$$f(x) = \varDelta \tag{1}$$

has one real and two different complex conjugate roots, and the real part of the complex roots is equal to a.

Proof. Since $(a, \varDelta) \in S_f$, then there exist complex conjugate numbers $z_1 = a + bi$ and $z_2 = a - bi$ such that $f(z_1) = \varDelta$ and $f(z_2) = \varDelta$.

Therefore, z_1, z_2 are the complex roots of the equation (1), so, the third root is real. Lemma 1 is proved.

Lemma 2. *Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ be a polynomial of the third order with real coefficients, $a_0 \neq 0$. Then the gluing set S_f of the polynomial $f(x)$ lies on the graph of the function $f_s(t)$, where $f_s(t) = f(-2t - a_1/a_0)$.*

The function $f_s(t)$ is said to be conjugate to the function $f(t)$.

Proof. Obviously, $f_s(t)$ is also a polynomial of the third order. Take an arbitrary point $(x_0, \Delta)\in S_f$ and construct the equation (1). By Lemma 1 , equation (1) has exactly one real root. Denote this real root $z_1(\Delta)$ and find other two complex roots z_2 and z_3. Obviously, numbers z_2 and z_3 are the roots of the polynomial of the second order

$$\frac{f(x)-\Delta}{x-z_1} = \frac{f(x)-f(z_1)}{x-z_1}$$
$$= \frac{a_0(x^3-z_1^3)+a_1(x^2-z_1^2)+a_2(x-z_1)}{x-z_1}$$
$$= a_0(x^2+xz_1+z_1^2)+a_1(x+z_1)+a_2 = A{\cdot}x^2+B{\cdot}x+C,$$

where

$$A=a_0, B=a_0z_1+a_1, C=a_0z_1^2+a_1z_1+a_2.$$

Equation

$$Az^2+Bz+C=0$$

has two roots $z_{2,3} = -\frac{B}{2A} \pm \frac{\sqrt{D}}{2A}$, where the discriminant $D = B^2-4{\cdot}A{\cdot}C$ depends on z_1 and hence on Δ.

Thus, equation (1) has three roots: one real z_1 and two complex conjugated ones

$$z_2 = -\frac{z_1}{2} - \frac{a_1}{2a_0} + \frac{\sqrt{D}}{2A}, z_3 = -\frac{z_1}{2} - \frac{a_1}{2a_0} - \frac{\sqrt{D}}{2A}.$$

By the definition of the set S_f, for the point $(x_0, \Delta)\in S_f$ there exist different complex conjugate numbers z_2^0 and z_3^0 such that

$$f(z_2^0) = f(z_3^0) = \Delta$$

and

$$Re z_2^0 = Re z_3^0 = x_0.$$

It follows that z_2^0 and z_3^0 are complex roots of equation (1) and, in particular, they differ from the real root z_1. Therefore, the roots z_2 and z_3 coincide with the numbers z_2^0 and z_3^0.

So, if $(x_0, \Delta)\in S_f$, then

$$x_0 = -\frac{z_1(\Delta)}{2} - \frac{a_1}{2a_0}. \tag{2}$$

The set S_f has the single-valued property: if (x_0, Δ_1) and (x_0, Δ_2) belong to S_f, then $\Delta_1 = \Delta_2$. Indeed, from (2) we obtain

$$z_1(\Delta_1) = z_1(\Delta_2) = -2x_0 - \frac{a_1}{a_0}.$$

Since $z_1(\Delta_1)$ and $z_1(\Delta_2)$ are the roots of the corresponding equation (1), then

$$\Delta_1 = f(z_1, (\Delta_2)) = f(z_1, (\Delta_2)) = \Delta_2.$$

Thus, the set S_f lies on the graph of a single-valued conjugate function $y = f_s(x)$, and $f_s(x_0) = \Delta$, that is,

$$f_s(-\frac{z_1(\Delta)}{2} - \frac{a_1}{2a_0}) = \Delta. \tag{3}$$

Replace $t = -\frac{z_1(\Delta)}{2} - \frac{a_1}{2a_0}$. Then $z_1(\Delta) = -2t - \frac{a_1}{a_0}$. Since z_1 is the root of (1), then

$$f(-2t - \frac{a_1}{a_0}) = f(z_1(\Delta)) = \Delta. \tag{4}$$

Comparing relations (3) and (4), we obtain $f_s(t) = f(-2t - \frac{a_1}{a_0})$. Lemma 2 is proved.

The graph of the conjugate function $f_s(x)$ is formed from the graph of the cubic parabola $f(x)$ in the following way. We compress the graph $f(x)$ twice with respect to the vertical axis passing through the inflection point and turn it around the same axis.

Lemma 3. *Let $f(x)$ be a cubic polynomial with real coefficients. Then the graph of the conjugate function $f_s(x)$ passes through points of local extrema $f(x)$, if they exist. In addition, the unique inflection points of the functions f and f_s coincide.*

Proof. Assume that x^* is a local extremum point of $f(x)$. One can find x^* from the equation $f'(x) = 0$ and verify the equality $f_s(x^*) = 0$ by a direct substitution. We simplify the calculations by changing the variable.

It is known that with parallel shifts the shape of the function graph does not change. Let us shift the graphs of the functions $f(x)$ and $f_s(x)$: horizontally by $r = -\frac{a_1}{3a_0}$ to the left using the substitution $x = u + r = u + (-\frac{a_1}{3a_0})$, and vertically down by a_3 .

We obtain $g(u) = f(u + (-\frac{a_1}{3a_0})) - a_3$. The function $g(u)$ is a cubic parabola, for which the inflection point coincides with the coordinates origin. This easily implies that $g(u) = A{\cdot}u^3 + C{\cdot}u$. Obviously, the function $g(u)$ is odd and passes through the coordinates origin.

It suffices to prove Lemma 3 for the function $g(u)$. We assume that $A > 0$ and suppose there are local extremum points for g. Then they have the form $u_{1,2} = \pm\sqrt{\frac{-C}{3A}}$, from which, in particular, $C < 0$. Substituting, for example, u_1 into the functions $g(u)$ and $g_s(u)$, we obtain

$$g(u_1) = A(\sqrt{-\frac{C}{3A}})^3 + C(\sqrt{-\frac{C}{3A}})$$
$$= -\frac{C}{3}\sqrt{-\frac{C}{3A}} + C\sqrt{-\frac{C}{3A}} = \frac{2}{3}C\sqrt{-\frac{C}{3A}},$$

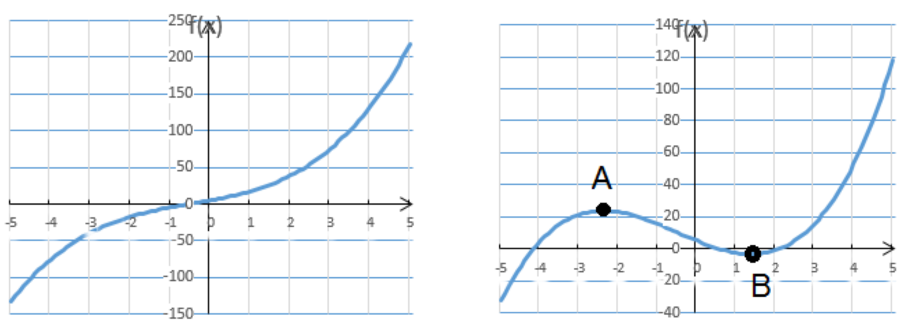
$$g_s(u_1) = g(-2u_1) = A(-2\sqrt{-\frac{C}{3A}})^3 + C(-2\sqrt{-\frac{C}{3A}})$$
$$= (-8) \cdot \frac{-C}{3}\sqrt{-\frac{C}{3A}} - 2{\cdot}C\sqrt{-\frac{C}{3A}} = \frac{2}{3}C\sqrt{-\frac{C}{3A}}.$$

Thus, the extremal points of the graph $g(u)$ lie on the graph of its conjugate function $g_s(u)$.

It is obvious that the inflection point of the function $g_s(u) = -8u^3 - 2Cu$ coincides with the inflection point of the function $g(u)$. Lemma 3 is proved.

3 Construction of a gluing set for a polynomial of order 3

Consider the polynomial $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, where $a_0 > 0$. The graph of a cubic polynomial $f(x)$ can be of two types (Fig. 1):



a) b)

Fig. 1. Two types of graphs of a cubic polynomial: a) without local extrema, b) with two local extrema

By Lemma 3, the graph of the conjugate function $f_s(x)$ passes through the extremum points and the inflection point of the function $f(x)$.

We construct the gluing set S_f, which by Lemma 2 is contained in the graph of the conjugate function $f_s(x)$.

The type of the cubic parabola (Fig. 1) depends on the sign of the discriminant of the cubic polynomial

$$D = 4(a_1^2 - 3a_0{\cdot}a_2).$$

Indeed, the abscissas of the points of local extrema of the function $f(x)$ are real roots of its derivative

$$f'(x) = 3a_0x^2 + 2a_1x + a_2.$$

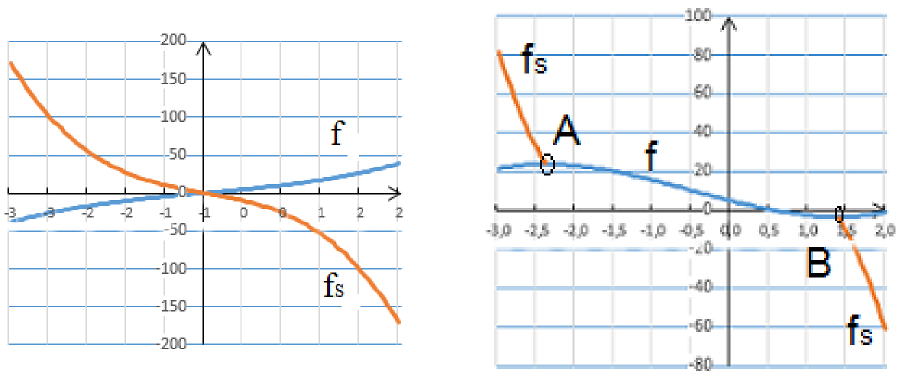
The discriminant of the quadratic equation $f'(x) = 0$, which determines the presence of real roots of $f(x)$, coincides with D.

If $D \leq 0$, then $f(x)$ does not have local extrema (Fig. 1a) and by Lemma 2 for any Δ the equation $f(x) = \Delta$ has one real and two complex roots. Therefore, gluing set S_f coincides with the entire graph of the conjugate function $f_s(x)$ (Fig. 1a).

Let $D > 0$. Then the graph of $f(x)$ has the form Fig 1b), where $A(x_1, y_1)$ and $B(x_2, y_2)$ are extreme points of the graph, and

$$y_1 > y_2, x_{1,2} = \frac{-2a_1 \pm \sqrt{D}}{6}$$

Take $\Delta_0 \in [y_2, y_1]$. Then the cubic equation $f(x) = \Delta_0$ has three real roots. If $\Delta_0 = y_2$ or $\Delta_0 = y_1$, then two of them are multiple. Therefore, by Lemma 2, for any x the point $(x, \Delta_0) \notin S_f$. Consequently, the horizontal line $y = \Delta_0$ passing through the point $(0, \Delta_0)$ does not intersect the set S_f. Thus, gluing set S_f is formed from the graph of the conjugate function $f_s(x)$ by removing the closed fragment of the graph from A to B (Fig 2b).



a)

b)

Fig. 2. Two types of gluing set for a third-order polynomial: a) without local extrema, b) with local extrema

Thus, on the same plane $\mathbb{R}^2$ three sets are constructed: the graph of the initial polynomial $f(x)$, the graph of the conjugate function $f_s(x)$, and the gluing set S_f, which is contained in second set.

4 Main results

We consider the cubic equation

$$f(x) \equiv x^3 + a_1x^2 + a_2x + a_3 = \Delta \tag{5}$$

for an arbitrary real Δ. We assume that $a_0 = 1$. Otherwise, the final results of this section can easily be modified by replacing the coefficients a_k by $\frac{a_k}{a_0}$ for $k = 1, 2, 3$. The real roots of equation (5) can be found from the graph of the original polynomial. We shall seek the complex roots of this equation in the form $x = a + bi, b \neq 0$.

Substituting x into (5), we obtain

$$\begin{gathered} f(a+bi) = (a+bi)^3 + a_1(a+bi)^2 + a_2(a+bi) + a_3 \\ = \left[a^3 + 3a^2bi - 3ab^2 - b^3\right] + a_1\left[a^2 + 2abi - b^2\right] + a_2\left[a + bi\right] + a_3 \\ = \left[a^3 - 3ab^2 + a_1a^2 - a_1b^2 + a_2a + a_3\right] + \left[3a^2b - b^3 + 2a_1ab + a_2b\right] i \\ = R(a,b) + I(a,b)\cdot i = \Delta + 0\cdot i. \end{gathered}$$

Equate to zero the imaginary part of $I(a, b)$ of the complex number $f(a+bi)$

$$3a^2b - b^3 + 2a_1ab + a_2b = 0.$$

Dividing by $b \neq 0$, we obtain

$$3a^2 - b^2 + 2a_1a + a_2 = 0, \tag{6}$$

whence

$$b^2 = 3a^2 + 2a_1a + a_2. \tag{7}$$

Transform the real part $R(a, b)$ of the number $f(a + bi)$

$$\begin{gathered} R(a,b) = [a^3 + a_1a^2 + a_2a + a_3] - [3a + a_1]\cdot b^2 \\ = [a^3 + a_1a^2 + a_2a + a_3] - [3a + a_1] \cdot [3a^2 + 2a_1a + a_2] \\ = a^3 + a_1a^2 + a_2a + a_3 - [9a^3 + 6a_1a^2 + 3a_2a + 3a_1a^2 + 2a_1^2a + a_1a_2] \\ = -8a^3 - 8a_1a^2 + (-2a_2 - 2a_1^2)a + (a_3 - a_1a_2) \\ = [-8a^3 - 12a^2a_1 - 6aa_1^2 - a_1^3] + a_1\cdot[4a^2 + 4aa_1 + a_1^2] + a_2\cdot[-2a - a_1] + a_3 \\ = (-2a - a_1)^3 + a_1(-2a - a_1)^2 + a_2(-2a - a_1) + a_3 = f(-2a - a_1) \\ = f(-2a - \frac{a_1}{a_0}) = f_s(a) = \Delta. \end{gathered}$$

Thus, it is proved

Lemma 4. *If the complex number $x = a + bi$ is a root of the equation* (5)*, then*

$$f_s(a) = \Delta. \tag{8}$$

From Fig. 2 we can see, that under the conditions of Lemma 4 the function f_s is invertible. Therefore, equality (8) can be written in the form

$$a = f_s^{-1}(\Delta). \tag{9}$$

Relation (9) gives a geometric algorithm of finding the real part a of the complex root $x = a + bi$ of the equation $f(x) = \Delta$. Now we show how to find the complex component b of the root x.

Definition 2. *The carrier set N_f of a polynomial $f(x)$ is defined to be the set of all complex numbers u with nonzero complex part, for which $f(u)$ is a real number, that is*

$$N_f = \{u \in \mathbb{C} : Im(u) \neq 0, f(u) \in \mathbb{R}\}. \tag{10}$$

On the real plane Oxy, we associate the axis Ox with the real part, and Oy with the imaginary part of the complex number $u = a + bi$.

Lemma 5. *The carrier set N_f of the polynomial $f(x)$ coincides with the graph of the multivalued mapping*

$$y = f_N(x) = \pm\sqrt{3x^2 + 2a_1{\cdot}x + a_2} = \pm\sqrt{f'(x)}.$$

The domain of $f_N(x)$ coincides with the projection of the set S_f onto the axis Ox.

Proof. Fix an arbitrary $\Delta \in \mathbb{R}$. Let $z_0 = a + bi, b \neq 0$, be a complex root of the equation $f(x) = \Delta$. Then, by Lemma 4, the real part of this root is $a = f_s^{-1}(\Delta)$, and the imaginary part satisfies relation (7). Therefore,

$$b = \pm\sqrt{3a^2 + 2a_1{\cdot}a + a_2}.$$

Lemma 4 is proved.

It is easy to see that the conjugate function $f_N(x)$ has two slope asymptotes

$$y = \sqrt{3}x + \frac{a_1}{\sqrt{3}}, y = -\sqrt{3}x - \frac{a_1}{\sqrt{3}}.$$

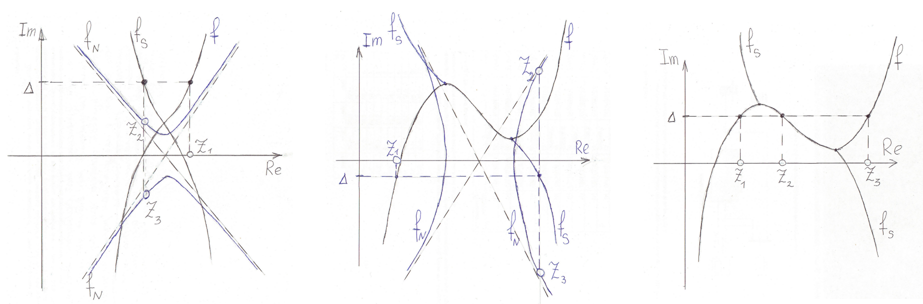
The slope of theese asymptotes is $\pm 60^o$ with respect to Ox.

5 Algorythm for the exact geometric search of all roots of a cubic equation

For simplicity of calculations, we shall assume that the elder coefficient of cubic equation $a_0 = 1$.

1. Choose an arbitrary $\Delta \in \mathbb{R}$.
2. Find the intersection of the graph $y = f(x)$ and the line $y = \Delta$. There are two possible cases:
   (a) The intersection consists of 3 points, of which two or three can coincide when the function graph and the line are touched. Then the abscissas z_1, z_2, z_3 of these points are obviously real roots (possibly multiples) of equation (1) (Fig. 3c). The algorithm is complete.
   (b) The intersection consists of one point. The abscissa z_1 of this point is the unique real root of equation (1). To find the two complex roots, go to step 3.

3. Find the intersection point of the line $y = \Delta$ and the gluing set S_f. This point is unique. Its abscissa a is the real part of the required pair of complex conjugate roots.
4. Find the intersection of the vertical line $x = a$ and the carrier set N_f, which lies on graph of f_N. This intersection consists of two different points. The ordinates of these points are equal to $+b$ and $-b$, where $b \geq 0$ and are imaginary components of the required pair of complex-conjugate roots of equation (1).
5. Thus, we have obtained two real numbers a and b, which yield complex conjugate roots of equation (1) (Fig. 3a, 3b). The algorithm is complete.



a) b) c)

Fig. 3. a) Always one real and two complex roots, b)one real and two complex roots, c)three real roots

5.1 Mathematical modeling

The results presented in this paper were obtained with the help of a previously developed application ([9], which allows one to plot the graphs of complex functions of a complex variable. The auxiliary gluing set S_f is the set of self-intersections for such graphs. The application is built in the MatLab system. To find complex roots of polynomials, the algorithm was also realized in MatLab. It allows us to trace the dynamics of complex roots when the free coefficient of the polynomial changes.

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