# Metric Right Propositional Neighborhood Logic with an Equivalence Relation

Pietro Sala[1]

Department of Computer Science, University of Verona, Strada le Grazie 15, Verona,
Verona `pietro.sala@univr.it`
http://profs.scienze.univr.it/ sala/

**Abstract.** In [13] Montanari et al. proved that the satisfiability problem for the fragments $\mathsf{A}$ and $\mathsf{A}\overline{\mathsf{A}}$ of Halpern and Shoham's modal logic of intervals extended with an equivalence relation $\sim$ over points, namely the logics $\mathsf{A}\sim$ and $\mathsf{A}\overline{\mathsf{A}}\sim$ respectively, is decidable and retains the same complexity (i.e., NEXPTIME-complete) of the same fragments deprived of the equivalence relation. In the same work the authors proved that the satisfiability problem for the logic $\mathsf{A}\overline{\mathsf{A}}\sim$ extended with metric constraints, namely $\mathsf{MPNL}\sim$, may be reduced to the reachability problem for Vector Addition Systems with States (VASS) [11] whose complexity upper bound is still unknown. In this paper we make a step forward in completing the picture by proving that the satisfiability problem for the missing fragment $\mathsf{A}\sim$ extended with metric constraints, namely $\mathsf{MRPNL}\sim$, is in 3EXPSPACE and it is EXPSPACE-hard.

**Keywords:** Interval Temporal Logic · Metric Constraints · Equivalence Relation · Complexity

## 1 Introduction

Extending decidable fragments of first-order logic such as the two-variable fragment with pre-interpreted relations is a topic that has been explored in the last two decades [8, 10]. Many of such extensions increase the expressivity of the logic while retaining the decidability of the satisfiability problem [22]. Other extensions push the limit too hard and the relative satisfiability problem turns out to be undecidable [6, 9]. Among series of technically beautiful results the extensions of two-variable fragment of first-order logic, namely $\mathsf{FO2}$, plays a crucial role. The satisfiability problem for this fragment and for its extension with a linear order relation, namely $\mathsf{FO2}[>]$, has been proved decidable for the most common kind of linear orders [20] including the reals [17].

In particular, $\mathsf{FO2}[>]$ has a very natural counterpart in the domain of Halpern and Shoham's modal logic of intervals [7], $\mathsf{HS}$ for short. It has been proved that the logic is equivalent to the fragment of $\mathsf{HS}$ that features both "meets" and "met by" modalities [3, 12], called propositional neighborhood logic $\mathsf{PNL}$ or, alternatively, $\mathsf{A}\overline{\mathsf{A}}$ from its two modalities $\mathsf{A}$ (i.e., "adjacent to the right") and $\overline{\mathsf{A}}$ (i.e., "adjacent to the left"). Let us consider now the logic $\mathsf{A}$, such (proper) fragment of PNL is referred to as *right propositional neighborhood logic*, RPNL for short, in the literature. RPNL may be seen as a future fragment of PNL, and thus of FO2[>], and presents many desirable properties that PNL does not. For instance, the extension of RPNL with the "begin" and "begun by" modalities, i.e., the logic $\mathsf{AB\overline{B}}$, retains the decidability of the satisfiability problem on the most common classes of linear orders [4, 16], while the same extension applied to $\mathsf{A\overline{A}}$, namely the logic $\mathsf{A\overline{A}B\overline{B}}$, turns out to be undecidable over infinite Dedekind-complete linear orders and on the finite or general linear orders it is proved to be NON-PRIMITIVE-HARD [14, 15]. When we add a pre-interpreted propositional variable $\sim$ that behaves like an equivalence relation the differences w.r.t. the decidability of the satisfiability problem between $\mathsf{AB\overline{B}}\sim$ and $\mathsf{A\overline{A}B\overline{B}}\sim$ are even more accentuated. $\mathsf{AB\overline{B}}\sim$ turns out to be decidable on the finite linear orders [18], even if we take into consideration the more general synthesis problem [19], while $\mathsf{A\overline{A}B\overline{B}}\sim$ turns out to be undecidable on the most common classes of linear orders.

In this paper we will prove the decidability for the satisfiability problem of MRPNL$\sim$ over finite linear orders by providing tighter complexity bounds. In particular, we prove that, when interpreted over finite linear orders, the satisfiability problem for the logic MRPNL$\sim$ belongs to the complexity class 3EXPSPACE. More precisely, we provide a reduction from MRPNL$\sim$ to the VASS coverability problem that will lead to an 3EXPSPACE decidability procedure. By giving this result we observe that MRPNL$\sim$ w.r.t. its super-fragment MPNL$\sim$ admits a known complexity bound, in fact in [2] a fragment of MPNL$\sim$ was proved capable of encoding 0-0 reachability problem for Vector Addition Systems with States (VASS) [11]. Moreover, we prove that the satisfiability problem for MRPNL$\sim$ is EXPSPACE-hard by means of a reduction to the coverability problem for Vector Addition Systems with States which was proved to be EXPSPACE-complete in [21]. We will observe that it turns out not so intuitive as in [2] and in [13] how satisfiability of MRPNL$\sim$ may be reduced to the coverability for VASS.

The paper is organized as follows. Section 2 provides a brief overview of the logic MRPNL$\sim$ as well as some basic definitions used throughout the paper. Section 3 describes the steps for proving that the satisfiability problem for MRPNL$\sim$ interpreted over finite linear orders belongs to the complexity class 3EXPSPACE. Section 4 describes the steps for proving that the satisfiability problem for MRPNL$\sim$ interpreted over finite linear orders is EXPSPACE-hard. Finally, Section 5 summarizes the results presented in this work and points out some interesting research directions that may be developed starting from what is proved here.

## 2 The logic MRPNL$\sim$

In this section we provide the syntax and the semantics of the *Metric Right Propositional Logic extended with an equivalence relation*, MRPNL$\sim$ for short, interpreted over finite linear orders. Let $\mathbb{N}$ be the set of natural numbers and let $N \subseteq \mathbb{N}$ a finite prefix of it. We denote by $I_N = \{[x, y] : x, y \in N, x \leq y\}$

the set of all possible intervals on $N$. MRPNL$\sim$ is a modal logic whose possible worlds are the elements of $I_N$ and the accessibility relation is given by the Allen interval relation "meets" [1] whose semantics is captured by the unary modality $\langle A\rangle$. Given two intervals $[x,y],[w,z]$ we say that $[x,y]$ *meets* $[w,z]$ if and only if $y=w$. Let $\mathcal{P}$ be a countable set of propositional variables whose elements are usually denoted with $p_1,p_2,\ldots$ and so on. In addition to $\mathcal{P}$, the language of MRPNL$\sim$ includes elements of $\mathbb{N}$ itself as special pre-interpreted propositional variables called *interval lengths* used for constraining the length of the intervals. More precisely, given $k\in\mathbb{N}$ we say that $k$ holds over an interval $[x,y]\in\mathbb{N}$ if and only if $y-x=k$ (i.e, $[x,y]$ has length $k$). Finally, the language of MRPNL$\sim$ features a special pre-interpreted propositional variable $\sim$ that represents an equivalence relation over $N$. In order to avoid clashes between variables, we assume $(\mathbb{N}\cup\{\sim\})\cap\mathcal{P}=\emptyset$. From now on we will denote MRPNL$\sim$ formulas with greek letters $\varphi,\psi,\ldots$ and so on. The syntax for MRPNL$\sim$ formulas is the following:

$$\varphi ::= p \mid n \mid \;\sim\; \mid \neg\psi \mid \psi\vee\psi \mid \langle A\rangle\psi \text{ with } p\in\mathcal{P} \text{ and } n\in\mathbb{N}$$

In the following we will use $\psi_1\wedge\psi_2$ as a shorthand for $\neg(\neg\psi_1\vee\neg\psi_2)$, $\psi_1\rightarrow\psi_2$ for $\neg\psi_1\vee\psi_2$, $\psi_1\leftrightarrow\psi_2$ for $(\psi_1\rightarrow\psi_2)\wedge(\psi_2\rightarrow\psi_1)$, $[A]\psi$ for $\neg\langle A\rangle\neg\psi$, $\not\sim$ for $\neg\sim$, $\top$ for $p\vee\neg p$, $\bot$ for $\neg\top$, and $[G]\psi$ (i.e., global modality) for $\psi\wedge[A]\psi\wedge[A][A]\psi$. Given an MRPNL$\sim$ formula $\varphi$ let us denote with $n_\varphi$ the maximum interval-length (if any) that appears in $\varphi$, if $\varphi$ does not feature any interval length we assume $n_\varphi=0$. Let $\Phi$ be the set of all possible MRPNL$\sim$ formulas. For the sake of complexity considerations that will be done in Section 3 and Section 4 we define a recursive function $\mathit{length}:\Phi\rightarrow\mathbb{N}$ as follows:

$$\mathit{length}(\varphi)=\begin{cases}1 & \text{if } \varphi\in\mathcal{P}\cup\mathbb{N}\cup\{\sim\}\\ 1+\mathit{length}(\psi) & \text{if } \varphi=\neg\psi \text{ or } \varphi=\langle A\rangle\psi\\ \mathit{length}(\psi_1)+\mathit{length}(\psi_2) & \text{if } \varphi=\psi_1\vee\psi_2\end{cases}$$

By means of function *length* we may define, given $\varphi\in\Phi$, the size of $\varphi$, denoted with $|\varphi|$, as $|\varphi|=\mathit{length}(\varphi)+\log_2(n_\varphi+2)$ (i.e, constants are supposed to be encoded in binary). A model for MRPNL$\sim$ is a pair $\mathbf{M}=(N,\mathcal{V},\sim_N)$ where $\mathcal{V}:I_N\rightarrow 2^{\mathcal{P}}$ and $\sim_N$ is an equivalence relation (i.e, a reflexive, symmetric, and transitive binary relation) over $N$. The semantics of MRPNL$\sim$ formulas $\varphi$ is given in terms of a pair $\mathbf{M},[x,y]$, where $\mathbf{M}=(N,\mathcal{V},\sim_N)$ is a finite model for MRPNL$\sim$ and $[x,y]\in I_N$, as follows:

- $\mathbf{M},[x,y]\models p$ with $p\in\mathcal{P}$ if and only if $p\in\mathcal{V}([x,y])$;
- $\mathbf{M},[x,y]\models n$ with $n\in\mathbb{N}$ if and only if $y-x=n$;
- $\mathbf{M},[x,y]\models\sim$ if and only if $x\sim_N y$;
- $\mathbf{M},[x,y]\models\neg\psi$ if and only if $\mathbf{M},[x,y]\not\models\psi$;
- $\mathbf{M},[x,y]\models\psi_1\vee\psi_2$ if and only if $\mathbf{M},[x,y]\models\psi_1$ or $\mathbf{M},[x,y]\models\psi_2$;
- $\mathbf{M},[x,y]\models\langle A\rangle\psi$ if and only if there exists $z\in\mathbb{N}$ s.t. $z\geq y$ and $\mathbf{M},[y,z]\models\psi$.

We say that a formula $\varphi$ is satisfiable if and only if there exists a finite model $\mathbf{M}=(N,\mathcal{V},\sim_N)$ for MRPNL$\sim$ and an interval $[x,y]\in I_N$ such that $\mathbf{M},[x,y]\models\varphi$. Then the (finite) satisfiability problem for MRPNL$\sim$ consists of, given an

MRPNL$\sim$ $\varphi$, deciding whether or not there exists a model $\mathbf{M} = (N, \mathcal{V}, \sim_N)$ and an interval $[x, y] \in I_N$ such that $\mathbf{M}, [x, y] \models \varphi$. It will turn out useful in Section 3 to impose the satisfiability of the main formula $\varphi$ to the first interval, namely $[0, 0]$. This is may be done safely using the following result.

**Theorem 1.** *For every* MRPNL$\sim$ *formula* $\varphi$ *there exists a model* $\mathbf{M} = (N, \mathcal{V}, \sim_N)$ *and an interval* $[x, y] \in I_N$ *such that* $\mathbf{M}, [x, y] \models \varphi$ *if and only if there exists a model* $\mathbf{M}' = (N', \mathcal{V}', \sim'_N)$ *with* $N' \subseteq N$ *such that* $\mathbf{M}', [0, 0] \models \langle A \rangle \varphi$.

Given $n \in \mathbb{N}$, we may impose that an interval has length greater than $n$ by means of the formula $\bigwedge_{0 \leq n' \leq n} \neg n'$. Moreover, we may force a propositional variable $n^{>}$ to hold on all and only the intervals with length greater than $n$ by means of the formula $[G](n^{>} \leftrightarrow \bigwedge_{0 \leq n' \leq n} \neg n')$. We will make use of such propositional variables in Section 4 where, for the sake of clarity, we will identify with $n^{\leq}$ the negation of $n^{>}$ (i.e., $n^{\leq}$ holds on all and only the intervals with length less or equal than $n$). These letters are mere syntactic sugar expressible in semantics of MRPNL$\sim$ provided above, however we prefer to assume them as pre-interpreted propositional variables like the ones in $\mathbb{N}$ or $\sim$. This is because the encoding $\bigwedge_{0 \leq n' \leq n} \neg n'$ is exponential in the size of $n$ and this will affect the LOGSPACE reduction we want to provide in Section 4. However, despite the fact that a linear encoding of $n^{>}$ in MRPNL$\sim$ (without embedding them in the semantics) is possible [5] we prefer to keep the things simple by adopting $n^{>}$ variables as pre-interpreted ones.

## 3 The MRPNL$\sim$ finite satisfiability problem

In this section we reduce the finite satisfiability problem for MRPNL$\sim$ to the coverability problem for Vector Addition System with States, VASS for short. The key idea behind this reduction is to use the VASS machinery to explore a (candidate) model for $\varphi$ starting from its maximum point and moving backwards. Since the $\langle A \rangle$ modality can only look forward, at every step the automaton consider the introduction of a new point $x$ at the beginning of such suffix. At this point if the automaton may guarantee the consistency conditions for $x$ (i.e., $[A]$-formulas) and the fulfilling of all the $\langle A \rangle$-requests associated to $x$ w.r.t. the suffix already explored then it may move to the next point or, if the input formula $\varphi$ is witnessed, it may successfully terminate. On the other hand, if the automaton cannot guarantee the aforementioned consistency/fulfilling conditions for $x$ then the computation fails.

A *vector addition system with states* (VASS) is a tuple $A = (Q, \Sigma, m, q_0, Q_f, \Delta)$ where $Q$ is a finite set of states, $\Sigma$ is a finite set of symbols called alphabeth, $m \in \mathbb{N}$, $q_0 \in Q$, $Q_f \subseteq Q$ is the set of final states, and $\Delta \subseteq Q \times \Sigma \times \mathbb{Z}^n \times Q$. Given a vector $v \in \mathbb{Z}^m$ and $1 \leq j \leq m$ we denote with $v[j]$ the value of the $j$-th component of $v$, moreover we define its size as $\sum_{1 \leq i \leq m} |v[i]|$. Given a vector

$v \in \mathbb{Z}^m$ it turns out to be useful to express the sum of the components until a given component $1 \leq j \leq m$ denoted as $|v|_j$ with $|v|_j = \sum\limits_{1 \leq i \leq j} |v[i]|$. Moreover, the sign of the component $j$ is defined as $sgn_j(v) = \begin{cases} + \text{ if } v[j] \geq 0 \\ - \text{ otherwise} \end{cases}$. Given a VASS $A = (Q, \Sigma, m, q_0, Q_f, \Delta$ we define its size $|A|$ as $|A| = 3 \cdot |Q| \cdot |\Sigma| \cdot |\Delta| \cdot m \cdot (\lceil \log_2(\max\{|v| : (q, \sigma, v, q') \in \Delta\} \rceil + 1)$.

A configuration $C$ of a VASS $A$ is a pair $C = (q, v)$ where $q \in Q$ and $v \in \mathbb{N}^m$. Let $\mathcal{C}_A$ be the set of all the possible configurations of $A$ (i.e., $\mathcal{C}_A = Q \times \mathbb{N}^m$). We define the transition relation over the configurations of $A$ as $\rightarrow_{.} \subseteq \mathcal{C}_A \times \Sigma \times \mathcal{C}_A$ where $(\overline{q}, \overline{v}) \rightarrow_{\overline{\sigma}} (\overline{q}', \overline{v}')$ if and only if there exists a transition $(q, \sigma, v, q')$ with $q = \overline{q}$, $q' = \overline{q}'$, $\sigma = \overline{\sigma}$ and $\overline{v} + v = \overline{v}'$. Given a word $w = \sigma_1 \ldots \sigma_k$ in $\Sigma^*$, let $\rightarrow^*_w$ be the reflexive and transitive closure of $\rightarrow_{.}$ using $w$, that is, $(\overline{q}, \overline{v}) \rightarrow^*_w (\overline{q}', \overline{v}')$ if and only if one of the following two conditions holds: (i) $w = \epsilon$ and $(\overline{q}, \overline{v}) = (\overline{q}', \overline{v}')$; (ii) $(\overline{q}, \overline{v}) \rightarrow_{\sigma_1} \ldots \rightarrow_{\sigma_k} (\overline{q}', \overline{v}')$. Given a VASS $A = (Q, \Sigma, m, q_0, F, \Delta)$ the *reachability problem* for $A$ consists of determining whether or not there exists a word $w \in \Sigma^*$ such that $(q_0, 0^m) \rightarrow^*_w (q, v)$ with $q \in F$. Given a VASS $A = (Q, \Sigma, m, q_0, F, \Delta)$ and a vector $v_f \in \mathbb{N}^m$ the *coverability problem* for the pair $(A, v_f)$ consists of determining whether or not there exists a word $w \in \Sigma^*$ such that $(q_0, 0^m) \rightarrow^*_w (q, v)$ with $q \in F$ and $v \geq v_f$. The size of a coverability problem $(A, v_f)$ is defined as $|(A, v_f)| = |A| \cdot m \cdot (\lceil \log_2(|v_f|) \rceil + 1)$.

Given a MRPNL∼ formula $\varphi$ let $\mathcal{C}\ell_\varphi$ be the set of all the sub-formulas of $\varphi$ together with their negations plus the formulas $\langle A \rangle \varphi$ and $[A] \neg \varphi$ and the set of sub-formulas $\{n, \neg n : 0 \leq n \leq n_\varphi\} \cup \{n^{>}_\varphi, n^{\leq}_\varphi\}$. Given an MRPNL∼ formula $\varphi$ an *$\varphi$-atom* $F \subseteq \mathcal{C}\ell_\varphi$ is a maximal consistent subset of the closure, that is:

- for every formula $\psi \in$ *closure* either $\psi \in F$ or $\neg \psi \in F$;
- for every formula $\psi_1 \vee \psi_2 \in \mathcal{C}\ell_\varphi$ we have $\psi_1 \vee \psi_2 \in F$ if and only $\psi_1 \in F$ or $\psi_2 \in F$;
- there exists at most one $0 \leq n \leq n_\varphi$ such that $n \in F$;
- if $0 \in F$ then for every $[A]\psi \in F$ we have $\psi \in F$.

We denote the set of all possible $\varphi$-atoms with $\mathcal{A}toms_\varphi$, let us observe that $|\mathcal{A}toms_\varphi| \leq 2^{|\varphi|+1} \cdot (n_\varphi + 2) \leq 2^{\mathcal{O}(|\varphi|)}$. Moreover, we define $\mathcal{A}toms^0_\varphi$ to be the set of atoms $F$ with $0 \in F$ (i.e., $\mathcal{A}toms^0_\varphi = \{F \in \mathcal{A}toms_\varphi : 0 \in F\}$) we call such atoms *zero-atoms*. Notice that, since the number of constants allowed to be positive in a zero-atom (as well as any $\varphi$-atom) is one (0 in this case), we have that $|\mathcal{A}toms^0_\varphi| \leq 2^{|\varphi|+1} \leq 2^{\mathcal{O}(|\varphi|)}$. We define the relation $\dot{\Rightarrow} \subseteq \mathcal{A}toms^0_\varphi \times (\mathcal{A}toms_\varphi \setminus \mathcal{A}toms^0_\varphi) \times \mathcal{A}toms^0_\varphi$ between pair of zero-atoms and atoms as follows. For every $F, F' \in \mathcal{A}toms^0_\varphi$ and for every $G \in \mathcal{A}toms_\varphi \setminus \mathcal{A}toms^0_\varphi$ we have $F \overset{G}{\Rightarrow} F'$ if and only if for every $\{\psi : [A]\psi \in F\} \cup \{[A]\psi \in F'\} \cup \{\langle A \rangle \psi \in F'\} \subseteq G$. Given a zero atom $F$ a *resolution* for $F$ is a minimal set $res_F = \{(G^0_1, G_1), \ldots, (G^0_k, G_k)\}$ of pairs in $\mathcal{A}toms^0_\varphi \times \mathcal{A}toms_\varphi \setminus \mathcal{A}toms^0_\varphi$ such that:

- for every $1 \leq k' \leq k$ we have $F \overset{G_{k'}}{\Rightarrow} G^0_{k'}$;
- for every $\langle A \rangle \psi \in F$ we have $\psi \in F$ or there exists $1 \leq k' \leq k$ such that $\psi \in G_{k'}$;
- for every $1 \leq n \leq n_\varphi$ there exists at most one $k'$ such that $n \in G_{k'}$.

For every $F \in \mathcal{A}toms^0_\varphi$ let $\mathcal{R}es_F$ be the set of all its possible resolutions for $F$. Notice that, since the minimality constraint forces $|\{\langle A\rangle\psi \in \mathcal{C}\ell_\varphi\}| \leq |\varphi|$, we have that $k \leq |\varphi|$. Let $\mathcal{R}es = \bigcup_{F\in\mathcal{A}toms^0_\varphi} \mathcal{R}es_F$ we have $|\mathcal{R}es| \leq |\mathcal{A}toms_\varphi|^{2|\varphi|}$.

A *$\varphi$-class-witness* is a multi-set $W$ whose elements are drawn from $\mathcal{A}toms^0_\varphi$. Moreover, every $\varphi$-class-witness satisfies the condition that for every $F \in \mathcal{A}toms^0_\varphi$ we have $W(F) \leq |\varphi| + n_\varphi + 1$. We denote with $\mathcal{W}itnesses_\varphi$ the set of all possible $\varphi$-class-witnesses. We define a *witness-union* $\sqcup_\varphi$ *operator* on $\mathcal{W}itnesses_\varphi$ as follows. Let $W, W'$, and $W''$ in $\mathcal{W}itnesses_\varphi$ we say that $W \sqcup_\varphi W' = W''$ if and only if for every $G \in \mathcal{A}toms^0_\varphi$ we have:

$$W''(G) = \begin{cases} W(G) + W'(G) \text{ if } W(G) + W'(G) \leq |\varphi| + n_\varphi + 1 \\ \quad |\varphi| + n_\varphi + 1 \quad \text{otherwise} \end{cases}$$

It is easy to see that $\mathcal{W}itnesses_\varphi$ is closed under $\sqcup_\varphi$, moreover $\mathcal{W}itnesses_\varphi$ is finite and its size is $|\mathcal{W}itnesses_\varphi| \leq (|\varphi|+n_\varphi+2)^{|\mathcal{A}toms^0_\varphi|} \leq 2^{\log(|\varphi|+n_\varphi+2)\cdot 2^{|\varphi|+1}} \leq 2^{2^{\mathcal{O}(|\varphi|)}}$. A *$\varphi$-state* is a tuple $\mathcal{S} = (\mathcal{W}, \sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi})$ where:

- $\mathcal{W}$ is a finite multi-set of $\varphi$-class-witnesses $\mathcal{W} = \{W_1, \ldots, W_m\}$;
- $\sim_{n_\varphi}$ is an equivalence relation over (a prefix of) $\{0, \ldots, n_\varphi\}$ (i.e., $\sim_{n_\varphi}$ is a partition of a prefix of $\{0, \ldots, n_\varphi\}$);
- $hor_{n_\varphi}$ is a partial function $hor_{n_\varphi} : \{0, \ldots, n_\varphi\} \rightarrow \mathcal{A}toms^0_\varphi$ called *horizon function* which for every $0 \leq n \leq n_\varphi$ such that for every $0 \leq n \leq n_\varphi$ if $hor_{n_\varphi}(n)$ is defined then $hor_{n_\varphi}(n')$ is defined for every $0 \leq n' \leq n$;
- $wit_{n_\varphi}$ is a partial function $wit_{n_\varphi} : \{0, \ldots, n_\varphi\} \rightarrow \mathcal{W}itnesses_\varphi$ called *witnesses function* such that for every $0 \leq n \leq n_\varphi$ if $wit_{n_\varphi}(n)$ is defined if and only if $hor_{n_\varphi}(n)$ is defined. Moreover for every $n \sim_{n_\varphi} n'$ for which both $wit_{n_\varphi}(n)$ and $wit_{n_\varphi}(n')$ are defined we have $wit_{n_\varphi}(n) = wit_{n_\varphi}(n')$;
- for every $0 \leq n \leq n_\varphi$ we have $\{hor_{n_\varphi}(n') : n' \sim_{n_\varphi} n\} \subseteq wit_{n_\varphi}(n)$.

In the following we will use the symbol $\perp$ to denote undefined values for functions $hor_{n_\varphi}$ and $wit_{n_\varphi}$. Let us observe that the above constraints imply that for every $0 \leq n \leq n_\varphi$ we have $hor_{n_\varphi}(n) \in wit_{n_\varphi}(n)$. We denote with $\mathcal{S}tates_\varphi$ the set of all possible $\varphi$-states. Given a $\varphi$-state $\mathcal{S} = (\mathcal{W}, \sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi})$ let $N^{\mathcal{S}}_{\sim}$ the subset $N^{\mathcal{S}}_{\sim_{n_\varphi}} \subseteq \{0, \ldots n_\varphi\}$ such that $n \in N^{\mathcal{S}}_{\sim_{n_\varphi}} \leftrightarrow n = \min[n]_{\sim_{n_\varphi}}$, thus $|N^{\mathcal{S}}_{\sim_{n_\varphi}}|$ is the index of $\sim_{n_\varphi}$. We denote with $\mathcal{W}_{\sim_{n_\varphi}}$ the multi-set $\{wit_{n_\varphi}(n) : n \in N^{\mathcal{S}}_{\sim_{n_\varphi}}\}$. Moreover, we denote $\mathcal{W}^{far}_{\sim_{n_\varphi}}$ the multi-set $\mathcal{W}^{far}_{\sim_{n_\varphi}} = \{wit_{n_\varphi}(n) \setminus \{hor_{n_\varphi}(n') : n' \sim_{n_\varphi} n : n \in N^{\mathcal{S}}_{\sim_{n_\varphi}}\}\}$. Each element $W_i$ in $\mathcal{W}^{far}_{\sim_{n_\varphi}}$ witnessed the zero-atoms for points that belongs to a class of a point $n$ in $\sim_{n_\varphi}$ but they are at distance greater than $n_\varphi$ w.r.t. the current horizon, this is why we call such points *far points*. Moreover, we define $\mathcal{W}^{far/0}_{\sim_{n_\varphi}} = \{wit_{n_\varphi}(n) \setminus \{hor_{n_\varphi}(n') : n' \sim_{n_\varphi} n : n \in N^{\mathcal{S}}_{\sim_{n_\varphi}} \setminus \{0\}\}\}$ (i.e., same as $\mathcal{W}^{far}_{\sim_{n_\varphi}}$ but without the multi-set for the far points in the equivalence class of 0).

Now, we define a reachability relation $\dot{\rightsquigarrow} \subseteq \mathcal{S}tates_\varphi \times \mathcal{A}toms^0_\varphi \times \mathcal{S}tates_\varphi$ between $\varphi$-states. Given two $\varphi$-states $\mathcal{S} = (\mathcal{W}, \sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi})$, $\mathcal{S}' = (\mathcal{W}', \sim'_{n_\varphi}, hor'_{n_\varphi}, wit'_{n_\varphi})$ and a zero atom $F \in \mathcal{A}toms^0_\varphi$ we have that $\mathcal{S} \overset{F}{\rightsquigarrow} \mathcal{S}'$ if and only

if there exists a resolution $res_F = \{(G_1^0, G_1), \ldots, (G_k^0, G_k)\}$ in $\mathcal{R}es_F$ such that:
**Local Consistency conditions:**

- *(LC1)* $hor'_{n_\varphi}(0) = F$ and for every $0 \leq n < n_\varphi$ we have $hor'_{n_\varphi}(n+1) = hor_{n_\varphi}(n)$;
- *(LC2)* for every $0 \leq n, n' < n_\varphi$ we have $n+1 \sim'_{n_\varphi} n'+1$ if and only if $n \sim_{n_\varphi} n'$;
- *(LC3)* for every $0 < n \leq n_\varphi$ such that $n \not\sim'_{n_\varphi} 0$ we have $wit'_{n_\varphi}(n) = wit_{n_\varphi}(n-1)$;
- *(LC4)* if there exists $0 < n \leq n_\varphi$ such $n \sim'_{n_\varphi} 0$ we have that $wit'_{n_\varphi}(0) = wit_{n_\varphi}(n-1) \sqcup_\varphi \{F\}$;
- *(LC5)* for every $0 < n \leq n_\varphi$ we have that if $n \sim'_{n_\varphi} 0$ (resp., $n \not\sim'_{n_\varphi} 0$) then there exists $G$ with $\{n, \sim\} \subseteq G$ (resp., $\{n, \not\sim\} \subseteq G$) such that $F \overset{G}{\Rightarrow} hor_{n_\varphi}(n)$;
- *(LC6)* for every $F' \in wit'_{n_\varphi}(0) \setminus \{hor'_{n_\varphi}(n) : n \sim'_{n_\varphi} 0\}$ we have that there exists $G$ such that $F \overset{G}{\Rightarrow} F'$ with $\{\sim, n_\varphi^>\} \subseteq G$;
- *(LC7)* for every $0 < n \leq n_\varphi$ such that $n \not\sim'_{n_\varphi} 0$ and for every $F' \in wit'_{n_\varphi}(n) \setminus \{hor'_{n_\varphi}(n') : n' \sim'_{n_\varphi} n\}$ there exists $G$ such that $F \overset{G}{\Rightarrow} F'$ with $\{\not\sim, n_\varphi^>\} \subseteq G$;

**Local Fulfilling conditions:**

- *(LF1)* for every $0 < n \leq n_\varphi$ such that there exists $1 \leq k' \leq k$ with $\{n\} \subseteq G_{k'}$ we have $hor'_{n_\varphi}(n) = G_{k'}^0$;
- *(LF2)* for every $0 < n \leq n_\varphi$ such that there exists $1 \leq k' \leq k$ with $\{n, \sim\} \subseteq G_{k'}$ (resp., $\{n, \not\sim\} \subseteq G_{k'}$ ) we have $n \sim'_{n_\varphi} 0$ (resp., $n \not\sim'_{n_\varphi} 0$);
- *(LF3)* $\{G_{k'}^0 : 1 \leq k' \leq k \wedge \{n_\varphi^>, \sim\} \subseteq G_{k'}\} \subseteq (wit'_{n_\varphi}(0) \setminus \{hor'_{n_\varphi}(n) : n \sim'_{n_\varphi} 0\})$;

**Global Consistency conditions:**

- *(GC1)* if there exists $0 < n \leq n_\varphi$ such that $n \sim'_{n_\varphi} 0$ we have that $\mathcal{W}' = (\mathcal{W} \setminus \{wit_{n_\varphi}(n-1)\}) \cup \{wit'_{n_\varphi}(0)\}$ (i.e., we have "updated" one $\varphi$-class-witness in $\mathcal{W}$ that contains a point in the last $n_\varphi$ points considered);
- *(GC2)* if for every $0 < n \leq n_\varphi$ such $n \not\sim'_{n_\varphi} 0$ and $wit'_{n_\varphi}(0) = \{F\}$ we have $\mathcal{W}' = \mathcal{W} \cup \{wit'_{n_\varphi}(0)\}$ (i.e., we have "inserted" one new $\varphi$-class-witness in $\mathcal{W}$);
- *(GC3)* if we have $|wit'_{n_\varphi}(0)| > 1$ and $n \not\sim'_{n_\varphi} 0$ for every $0 < n \leq n_\varphi$ then there exists $W \in \mathcal{W} \setminus \mathcal{W}_{\sim_{n_\varphi}}$ such that $\mathcal{W}' = (\mathcal{W} \setminus \{W\}) \cup \{W \sqcup_\varphi \{F\}\}$ (i.e., we have "updated" one $\varphi$-class-witness in $\mathcal{W}$ that does not contain a point in the last $n_\varphi$ points considered);
- *(GC4)* for every $W \in \mathcal{W}' \setminus \mathcal{W}'_{\sim'_{n_\varphi}}$ and for every $F' \in W$ we have that there exists $G$ such that $F \overset{G}{\Rightarrow} F'$ with $\{\not\sim, n_\varphi^>\} \subseteq G$;

**Global Fulfilling condition:**

- *(GF1)* $\{G_{k'}^0 : 1 \leq k' \leq k \wedge \{n_\varphi^>, \not\sim\} \subseteq G_{k'}\} \subseteq ((\mathcal{W}' \setminus \mathcal{W}'_{\sim'_{n_\varphi}}) \cup \mathcal{W}^{far/0}_{\sim_{n_\varphi}})$.

Given a word $wf = F_1 \ldots F_k$ on the alphabeth $(\mathcal{A}toms_\varphi^0)^*$, let $\overset{wf}{\rightsquigarrow}_*$ be the reflexive and transitive closure of $\dot{\rightsquigarrow}$ using $wf$, that is, $\mathcal{S} \overset{wf}{\rightsquigarrow}_* \mathcal{S}'$ if and only if one of the following two conditions holds: (i) $wf = \epsilon$ and $\mathcal{S} = \mathcal{S}'$; (ii) $\mathcal{S} \overset{F_1}{\rightsquigarrow} \ldots \overset{F_k}{\rightsquigarrow} \mathcal{S}'$. We may prove the following result on the relation $\dot{\rightsquigarrow}$.

**Theorem 2.** *Given an* MRPNL∼ *formula $\varphi$ we have that $\varphi$ is finitely satisfiable if and only if there exists a word of zero-atoms $wf = F_1 \ldots F_r \in (\mathcal{A}toms_\varphi^0)^*$ and* *two $\varphi$-states $\mathcal{S}_0, \mathcal{S}_f \in \mathcal{S}_M$ such that $\langle A \rangle \varphi \in F_r$, $\mathcal{W}^0 = \emptyset$, for every $0 \leq n \leq n_\varphi$ we have $hor^0_{n_\varphi}(n) = wit^0_{n_\varphi}(n) = \bot$ , and $\mathcal{S}_0 \overset{wf}{\rightsquigarrow}_* \mathcal{S}_f$.*

Before encoding $\overset{\cdot}{\rightsquigarrow}$ into a VASS we need some additional definitions in order to deal with purely technical details in such encoding. A *distinct function* is any function $dist_\varphi : \mathcal{A}toms^0_\varphi \rightarrow \{0, unique, distinct\}$. Let $\mathcal{D}ist_\varphi$ the set of all possible distinct functions, it is easy to prove that $|\mathcal{D}ist_\varphi| = 3^{|\mathcal{A}toms^0_\varphi|} \leq 2^{2^{\mathcal{O}(|\varphi|)}}$. Moreover, we say that a multi-set $\mathcal{W}_{GF}$ whose elements are drawn from $\mathcal{W}itnesses_\varphi$ is a *global fulfilling set* if and only if $|\mathcal{W}_{GF}| \leq |\varphi|$. Let $\mathcal{G}\mathcal{F}ull$ the set of all possible global fulfilling sets, it is easy to prove that $|\mathcal{G}\mathcal{F}ull| = (|\varphi| + n_\varphi + 2)^{|\mathcal{A}toms^0_\varphi|} \leq 2^{2^{\mathcal{O}(|\varphi|)}}$. Let $Eq^{\sim}_{n_\varphi}$ be the set of all possible $\sim_{n_\varphi}$, that is, all the possible equivalence relations (i.e., partitions) on sets with cardinalities ranging from 1 to $n_\varphi + 1$ plus the empty set which will be our starting $\sim_{n_\varphi}$. Set $Eq^{\sim}_{n_\varphi}$ is finite and we have $|Eq^{\sim}_{n_\varphi}| \leq 2^{|\varphi| \log(|\varphi|)} + 1 \leq 2^{\mathcal{O}(|\varphi| \log(|\varphi|))}$. Let $\mathcal{H}or_\varphi$ (resp., $\mathcal{W}it_\varphi$) the set of all possible horizon (resp., witness) functions, we have that $|\mathcal{H}or_\varphi| \leq (|\mathcal{A}toms^0_\varphi| + 1)^{n_\varphi + 1} \leq 2^{\mathcal{O}(|\varphi|^2)}$ (resp., $\mathcal{W}it_\varphi \leq (|\mathcal{W}itnesses_\varphi| + 1)^{n_\varphi + 1} \leq 2^{2^{|\varphi|}}$).

Now we have all the ingredients for defining a VASS $A_\varphi$ that encodes the relation $\overset{\cdot}{\rightsquigarrow}$. We begin by pointing out that the vectors $w$ in the computation of $A_\varphi$ are indexed with the elements of $\mathcal{W}itnesses_\varphi$ since they represent a portion of multi-sets in $\mathcal{W}$. Given two multi-sets $\mathcal{W}, \mathcal{W}'$ whose elements are drawn from $\mathcal{W}itnesses_\varphi$, we will define the difference $\mathcal{W} - \mathcal{W}'$ operation, which is different from the multi-set operation $\mathcal{W} \setminus \mathcal{W}'$, as the operation that returns the vector $v \in \mathbb{Z}^{|\mathcal{W}itnesses_\varphi|}$ (indexed on the elements of $\mathcal{W}itnesses_\varphi$) which is such that $v[W] = \mathcal{W}(W) - \mathcal{W}'(W)$ for every $W \in \mathcal{W}itnesses_\varphi$.

Given an input $\mathsf{MRPNL}\!\sim$ formula $\varphi$ we define the VASS $A_\varphi = (Q, \Sigma, m, Q_f, \Delta)$ where $Q = Eq^{\sim}_{n_\varphi} \times \mathcal{H}or_\varphi \times \mathcal{W}it_\varphi \times \mathcal{D}ist_\varphi \times \mathcal{G}\mathcal{F}ull$, $\Sigma = \mathcal{A}toms^0_\varphi \times \mathcal{R}es \times \mathcal{W}itnesses_\varphi$, $m = |\mathcal{W}itnesses_\varphi|$, $q_0 = (\emptyset, \bot, \bot, 0, \emptyset)$, $Q_f = \{(\sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi}, dist_\varphi, \mathcal{W}_{GF}) \in Q : \langle A \rangle \varphi \in hor_{n_\varphi}(0)\}$. Moreover we have $((\sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi}, dist_\varphi, \mathcal{W}_{GF}), (F, \mathcal{R}es_F, \overline{W}), w, (\sim'_{n_\varphi}, hor'_{n_\varphi}, wit'_{n_\varphi}, gc', dist'_\varphi, \mathcal{W}'_{GF})) \in \Delta$ if and only if the following conditions hold (let $res_F = \{(G_1, G^0_1), \ldots, (G_k, G^0_k)\}$):

1. $\overline{W} \sqcup_\varphi \{F\} = wit'_{n_\varphi}(0)$ and both **local consistency conditions** and **local fulfilling conditions** are satisfied by $hor_{n_\varphi}, wit_{n_\varphi}, F, res_F, hor'_{n_\varphi}$ and $wit'_{n_\varphi}$;
2. let $\mathcal{W}_{out} = \begin{cases} \mathcal{W}_{GF} & \text{if } n_\varphi \neq \min[n_\varphi]_{\sim_{n_\varphi}} \\ \mathcal{W}_{GF} \cup \{wit_{n_\varphi}(n_\varphi) \sqcup_\varphi hor_{n_\varphi}(n_\varphi)\} & \text{otherwise} \end{cases}$
   then we have $w = \begin{cases} \mathcal{W}_{out} - \mathcal{W}'_{GF} & \text{if } 0 \neq \max[0]_{\sim'_{n_\varphi}} \\ \mathcal{W}_{out} - (\mathcal{W}'_{GF} \cup \{\overline{W}\}) & \text{otherwise} \end{cases}$ ;
3. for every $F' \in \mathcal{A}toms^0_\varphi$ if $dist'_\varphi(F') = distinct$ or $dist'_\varphi(F') = unique \wedge F' \notin (wit'_{n_\varphi}(0) \setminus \{hor_{n_\varphi}(n) : n \sim'_{n_\varphi} 0\})$ then there exists $G$ with $\{\not\sim, n^{>}_\varphi\} \in G$ with $F \overset{G}{\Rightarrow} F'$;

4. for every $F' \in \mathcal{A}toms^0_\varphi$:

$$dist'_\varphi(F') = \begin{cases} distinct & \text{if } hor_{n_\varphi}(n_\varphi) = F',\ dist_\varphi(F') = unique, \\ & \text{and } F' \notin (wit_{n_\varphi}(n_\varphi) \setminus \{hor_{n_\varphi}(n) : n \sim_{n_\varphi} n_\varphi\}) \\ unique & \text{if } hor_{n_\varphi}(n_\varphi) = F' \text{ and } dist_\varphi(F') = 0 \\ dist_\varphi(F') & \text{otherwise} \end{cases} \quad ;$$

5. $\{G^{k'} : 1 \leq k' \leq k, \{\not\sim, n^{>}_\varphi\} \subseteq G^{k'}\} \subseteq (\bigcup_{W \in \mathcal{W}'_{GF}} W \cup \bigcup_{W \in \mathcal{W}^{far/0'}_{\sim'_{n_\varphi}}} W)$.

The following result basically guarantees the soundness and completeness of the above reduction.

**Theorem 3.** *For every pair of configurations* $C = ((\sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi}, dist_\varphi, \mathcal{W}_{GF}), \mathcal{W})$, $C' = ((\sim'_{n_\varphi}, hor'_{n_\varphi}, wit'_{n_\varphi}, dist'_\varphi, \mathcal{W}'_{GF}), \mathcal{W}')$ *and every symbol* $\sigma = (F, \mathcal{R}es_F, \overline{W}) \in \Sigma$ *we have that* $C \rightarrow_\sigma C'$ *in* $A_\varphi$ *if and only if* $(\mathcal{W}_{\sim_{n_\varphi}} \cup \mathcal{W}_{GF} \cup \mathcal{W}, \sim_{n_\varphi}, hor_{n_\varphi}, wit_{n_\varphi}) \overset{F}{\rightsquigarrow} (\mathcal{W}'_{\sim'_{n_\varphi}} \cup \mathcal{W}'_{GF} \cup \mathcal{W}', \sim'_{n_\varphi}, hor'_{n_\varphi}, wit'_{n_\varphi})$.

Let us consider now the size of $A_\varphi$. According to the complexity bounds given so far we have $|Q| \leq 2^{2^{\mathcal{O}(|\varphi|)}}$, $|\Sigma| \leq 2^{2^{\mathcal{O}(|\varphi|)}}$, $|\Delta| \leq 2^{2^{\mathcal{O}(|\varphi|)}}$, and $(\lceil \log_2(\max\{|v| : (q, \sigma, v, q') \in \Delta\} \rceil + 1) \leq 2\lceil \log_2 |\varphi| \rceil$. Finally we have that:

$$|A_\varphi| \leq 2^{2^{\mathcal{O}(|\varphi|)}}$$

which is doubly exponential in the size of $|\varphi|$. We may use the VASS coverability problem on $A_\varphi$ to see if $(\emptyset, \bot, \bot) \overset{w}{\rightsquigarrow}_* (\mathcal{W}, hor_{n_\varphi}, wit_{n_\varphi})$ with $\langle A \rangle \varphi \in hor_{n_\varphi}(0)$. Then, since the VASS coverability problem is EXPSPACE-complete [21], we conclude this section with the following result.

**Theorem 4.** *The finite satisfiability problem for* MRPNL∼ *belongs to the complexity class 3EXPSPACE.*

## 4 EXPSPACE-hardness of MRPNL∼

In this section we provide a LOGSPACE reduction from the coverability problem for Vector Addition Systems with States to the (finite) satisfiability problem for MRPNL∼, thus proving that the latter is EXPSPACE-hard. Given a VASS $A = (Q, \Sigma, m, q_0, Q_f, \Delta)$ and a vector $v_f \in \mathbb{N}^m$ we write a formula $\varphi_A$, with $|\varphi_A| = \mathcal{O}(|A| + |v_f|)$ which is satisfiable if and only if $(A, v_f)$ is a positive instance of the coverability problem for VASS. The idea behind $\varphi$ is to encode all the possible computations of $A$ (if any) beginning in $(q_0, 0)$ and finishing in a configurations $(q, v)$ with $v \geq v_f$. As a matter of fact, we will encode a slightly different problem. Let $\hat{q}_f \notin Q_f$ be a fresh state we define $A' = (Q', \Sigma, m, q_0, \{\hat{q}_f\}, \Delta')$ where $Q' = Q \cup \{\hat{q}_f\}$ and $\Delta' = \Delta \cup \{(q, \sigma, -v_f, \hat{q}_f) : q \in Q_f, \sigma \in \Sigma\}$. It is easy to prove that $(A, v_f)$ is a positive instance of the coverability problem for VASS if and only if $A'$ is a positive instance of the reachability problem. Let us begin by encoding the sequence of configurations of $A'$. The computation is encoded backwards with respect to the temporal domain. In order to do this we make use of $|Q|$ propositional variables $q_i \in Q$ for encoding states and $2 \cdot m$ propositional variables $C^+ = \{c_1^+, \ldots, c_m^+\}$ and $C^- = \{c_1^-, \ldots, c_m^-\}$, for encoding counters in the computation of $A$, we impose the uniqueness over each point in the model for all these variables by means of the following formulas:

$$\psi_! = [G]\left(0 \leftrightarrow \bigwedge_{p \in Q \cup C^+ \cup C^-}\left(p \rightarrow \bigwedge_{p' \in ((Q \cup C^+ \cup C^-) \setminus \{p\})} \neg p'\right)\right)$$

$$\psi_\exists = [G]\left(0 \leftrightarrow \bigvee_{p \in Q \cup C^+ \cup C^-} p\right)$$

In order to simplify the encoding we introduce $|\mathit{\Delta}|$, auxiliary variables, for better encoding the transition relation. Let $\mathit{\Delta} = \{(q^1, \sigma^1, v^1, \overline{q}^1), \ldots, (q^h, \sigma^h, v^h, \overline{q}^h)\}$ then we introduce $h$ auxiliary variables $\mathit{\Delta} = \{\delta_1, \ldots, \delta_h\}$, constrained as follows:

$$\psi_!^{\mathit{\Delta}} = [G]\left(\bigwedge_{\delta_i \in \mathit{\Delta}}\left(\delta_i \rightarrow \overline{q}^i \wedge \bigwedge_{\delta_j \in (\mathit{\Delta} \setminus \{\delta_j\})} \neg \delta'_j\right)\right)$$

$$\psi_Q^{\mathit{\Delta}} = [G]\left(\bigwedge_{q \in Q}\left(q \wedge \langle A \rangle \top \rightarrow \bigvee_{\delta_i \in \{\delta_j \in \mathit{\Delta} :\ \overline{q}^j = q\}} \delta_i\right)\right)$$

Basically any state but the first one in a sequence of configurations is labelled with the transition in $\mathit{\Delta}$ that has generated it. Each transition $\delta_i \in \mathit{\Delta}$ is encoded by first putting a point labelled with $\overline{q}^i$ (i.e., the target state) on the current point $x$ and then, according to $v^i$, a sequence of $|v^i|$ points labelled with components $c_j^*$ with $* \in \{-, +\}$ and $1 \leq j \leq m$, finally, point $x + |v| + 1$ is labelled with $q^i$ (i.e., the source state). For every $\delta_i \in \mathit{\Delta}$ this is done by means of the following formula:

$$\psi^{\delta_i} = [G]\left(\delta^i \rightarrow \begin{array}{c} \langle A \rangle(|v^i| + 1 \wedge \langle A \rangle q^i) \wedge [A](0^> \wedge |v^i|_1^{\leq} \rightarrow \langle A \rangle(0 \wedge c_1^{sgn_1(v^i)})) \wedge \\ \bigwedge_{1 < j \leq m} [A](|v^i|_{j-1}^{>} \wedge |v^i|_j^{\leq} \rightarrow \langle A \rangle(0 \wedge c_j^{sgn_j(v^i)}))) \end{array}\right)$$

Notice that in case $v^i[j] = 0$ then both conditions $|v^i|_{j-1}^{>} \wedge |v^i|_j^{\leq}$ and $0^> \wedge |v^i|_1^{\leq}$ are false and thus the case where components of vector $v^i$ with value 0 is treated in a transparent way. Now we have to deal with the problem of ensuring that each counter is correctly incremented/decremented along the computation. Since we are in the coverability problem the only thing that we must ensure is that a decrement of a component cannot happen if there is not a corresponding increment "before" in the computation. Let us recall that in our encoding "before" means in the future with respect to the temporal domain. At this point, the equivalence relation $\sim$ comes into play by mapping, for each $1 \leq j \leq m$, each point labelled with $c_j^-$ into a point labelled $c_j^+$ in its "future". It is worth noticing that we must guarantee that such mapping is injective. This is why the properties of $\sim$ are crucial. The aforementioned mapping is achieved by means of the following formula:

$$\psi^{inj} = [G]\left(\bigwedge_{1\leq j\leq m} \left(c_j^- \rightarrow [A](\neg 0 \wedge \sim \rightarrow [A](\neg 0 \rightarrow \not\sim)) \wedge \langle A\rangle(\sim \wedge \langle A\rangle c_j^+)\right)\right)$$

Finally, we just force our model to have a $q_0$-labelled point as its maximum and a in $\hat{q}_f$-labelled point as its minimum. This is achieved by means of the following formula:

$$\psi_{\min}^{\max} = 0 \wedge \hat{q}_f \wedge \langle A\rangle([A]0 \wedge \langle A\rangle q_0)$$

The formula $\varphi^A$ is defined as $\varphi_A = \psi_! \wedge \psi_\exists \wedge \psi_!^\Delta \wedge \psi_Q^\Delta \wedge \bigwedge_{\delta_i \in \Delta} \psi^{\delta_i} \wedge \psi^{inj} \wedge \psi_{\min}^{\max}$

It is long, tedious and trivial to prove that $\varphi_A$ is finitely satisfiable if and only if there $\hat{q}_f$ is reachable from $(q_0, 0^m)$ in $A'$. Let us notice that, since the constants are expressed in binary, we have $|\varphi| = \mathcal{O}(|\mathcal{A}'|)$. It is easy to prove that this translation may be done in logarithmic space and thus we have the following result.

**Theorem 5.** *The finite satisfiability problem for* MRPNL∼ *is EXPSPACE-hard.*

## 5 Conclusions

In this paper we proved that the satisfiability problem for the logic MRPNL∼ is decidable over finite linear orders with elementary complexity. More precisely, we proved that such problem belongs to the complexity class 3EXPSPACE and it is EXPSPACE-hard. In the not-so-distant future we plan to address the exact complexity of this problem. Moreover, since MRPNL∼ represents a weaker but, in principle, computationally more treatable version of its super-fragment MPNL∼ we plan to study the properties that can be captured by MRPNL∼ as well as properties that separate the two fragments (i.e., properties expressible in MPNL∼ that cannot be expressed in MPNL∼). Finally, we plan to study the decidability/complexity of satisfiability problem for MRPNL∼ when it is interpreted over $\mathbb{N}$.

## References

1. Allen, J.F.: Maintaining knowledge about temporal intervals. Commun. ACM **26**(11), 832–843 (Nov 1983). https://doi.org/10.1145/182.358434
2. Bojańczyk, M., David, C., Muscholl, A., Schwentick, T., Segoufin, L.: Two-variable logic on data words. ACM Trans. Comput. Log. **12**(4), 27:1–27:26 (2011). https://doi.org/10.1145/1970398.1970403
3. Bresolin, D., Goranko, V., Montanari, A., Sciavicco, G.: Propositional interval neighborhood logics: Expressiveness, decidability, and undecidable extensions. Ann. Pure Appl. Logic **161**(3), 289–304 (2009). https://doi.org/10.1016/j.apal.2009.07.003

4. Bresolin, D., Montanari, A., Sala, P., Sciavicco, G.: What's decidable about halpern and shoham's interval logic? the maximal fragment ABBL. In: LICS 2011. pp. 387–396 (2011). https://doi.org/10.1109/LICS.2011.35
5. Bresolin, D., Montanari, A., Sala, P., Sciavicco, G.: Optimal decision procedures for MPNL over finite structures, the natural numbers, and the integers. Theor. Comput. Sci. **493**, 98–115 (2013). https://doi.org/10.1016/j.tcs.2012.10.043
6. Grädel, E., Otto, M., Rosen, E.: Undecidability results on two-variable logics. In: STACS 97. pp. 249–260 (1997). https://doi.org/10.1007/BFb0023464
7. Halpern, J.Y., Shoham, Y.: A propositional modal logic of time intervals. J. ACM **38**(4), 935–962 (Oct 1991). https://doi.org/10.1145/115234.115351
8. Kieronski, E.: Results on the guarded fragment with equivalence or transitive relations. In: CSL 2005. pp. 309–324 (2005). https://doi.org/10.1007/11538363_22
9. Kieronski, E., Otto, M.: Small substructures and decidability issues for first-order logic with two variables. In: (LICS 2005). pp. 448–457 (2005). https://doi.org/10.1109/LICS.2005.49
10. Kieronski, E., Pratt-Hartmann, I., Tendera, L.: Equivalence closure in the two-variable guarded fragment. J. Log. Comput. **27**(4), 999–1021 (2017). https://doi.org/10.1093/logcom/exv075, https://doi.org/10.1093/logcom/exv075
11. Leroux, J.: Vector addition system reachability problem: a short self-contained proof. In: POPL 2011. pp. 307–316 (2011). https://doi.org/10.1145/1926385.1926421
12. Montanari, A., Pazzaglia, M., Sala, P.: Metric propositional neighborhood logic with an equivalence relation. In: TIME 2014. pp. 49–58 (2014). https://doi.org/10.1109/TIME.2014.26
13. Montanari, A., Pazzaglia, M., Sala, P.: Metric propositional neighborhood logic with an equivalence relation. Acta Inf. **53**(6-8), 621–648 (2016). https://doi.org/10.1007/s00236-016-0256-3
14. Montanari, A., Puppis, G., Sala, P.: Maximal decidable fragments of halpern and shoham's modal logic of intervals. In: ICALP 2010. pp. 345–356 (2010). https://doi.org/10.1007/978-3-642-14162-1_29
15. Montanari, A., Puppis, G., Sala, P.: Decidability of the interval temporal logic $\mathsf{A\bar{A}B\bar{B}}$ over the rationals. In: MFCS 2014. pp. 451–463 (2014). https://doi.org/10.1007/978-3-662-44522-8_38
16. Montanari, A., Puppis, G., Sala, P., Sciavicco, G.: Decidability of the interval temporal logic ABB over the natural numbers. In: STACS. pp. 597–608 (2010). https://doi.org/10.4230/LIPIcs.STACS.2010.2488
17. Montanari, A., Sala, P.: An optimal tableau system for the logic of temporal neighborhood over the reals. In: TIME 2012. pp. 39–46 (2012). https://doi.org/10.1109/TIME.2012.18
18. Montanari, A., Sala, P.: Adding an equivalence relation to the interval logic ABB: complexity and expressiveness. In: LICS 2013. pp. 193–202 (2013). https://doi.org/10.1109/LICS.2013.25
19. Montanari, A., Sala, P.: Interval-based synthesis. In: GandALF 2014. pp. 102–115 (2014). https://doi.org/10.4204/EPTCS.161.11
20. Otto, M.: Two variable first-order logic over ordered domains. J. Symb. Log. **66**(2), 685–702 (2001). https://doi.org/10.2307/2695037, https://doi.org/10.2307/2695037
21. Rackoff, C.: The covering and boundedness problems for vector addition systems. Theoretical Computer Science **6**(2), 223–231 (1978)
22. Szwast, W., Tendera, L.: Foˆ2 with one transitive relation is decidable. In: STACS 2013. pp. 317–328 (2013). https://doi.org/10.4230/LIPIcs.STACS.2013.317