Power (Set) $\mathcal{ALC}$

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Abstract. We explore the relationships between Description Logics and Set Theory. The study is carried on using, on the set-theoretic side, a very rudimentary axiomatic set theory Ω, consisting of only four axioms characterizing binary union, set difference, inclusion, and the power-set. The approach is then completed defining $\mathcal{ALC}^{\Omega}$, an extension of $\mathcal{ALC}$ in which concepts are naturally interpreted as sets living in Ω-models. In $\mathcal{ALC}^{\Omega}$ not only membership between concepts is allowed—even admitting circularity—but also the power-set construct is exploited to add metamodeling capabilities. We conclude providing a polynomial translation of $\mathcal{ALC}^{\Omega}$ in $\mathcal{ALCOI}$ and proving its basic traits, among which the validity of the finite model property.

1 Introduction

Concept and concept constructors in Description Logics (DLs) allow to manage information built-up and stored as collection of elements of a given domain. In this paper we would like to take the above statement seriously and put forward a DL doubly linked with a (very simple, axiomatic) set theory. Such a theory will be suitable to manipulate concepts (also called classes in OWL [20]) as first-class citizens, in the sense that it will allow the possibility to have concepts as instances (a.k.a. elements) of other concepts. Actually, the idea of enhancing the language of description logics with statements of the form $C \in D$, with C and D concepts is not new, as assertions of the form $D(A)$, with A a concept name, are allowed in OWL-Full [20]. Here, while we do not consider roles, i.e. relations among individuals (also called properties in OWL), as possible instances of concepts, we would like to push the approach a little forward, allowing not only the possibility of stating that an arbitrary concept C can be thought as an instance of another one ($C \in D$), but also opening up our view along two further directions:

1. the possibility of having, as a special case, a concept as an instance of itself: $C \in C$[1];
2. the possibility of talking about *all possible* sub-concepts of a given concept, adding a power-set construct $\texttt{Pow}(C)$.

In order to realize our plan we introduce a DL, to be dubbed $\mathcal{ALC}^{\Omega}$, whose two parents are $\mathcal{ALC}$ and a rudimentary (finitely axiomatized) set theory Ω.

[1] Self membership is allowed for concept names in [16], by assertions of the form $a(a)$

For instance, considering an example taken from [22, 16], using membership axioms, we can represent the fact that eagles are in the red list of endangered species, by the axiom $Eagle \in RedListSpecies$ and that Harry is an eagle, by the assertion $Eagle(harry)$. We could further consider a concept $notModifiableList$, consisting of those lists that cannot be modified (if not by, say, a specifically enforced law) and, for example, it would be reasonable to ask $RedListSpecies \in notModifiableList$ but, more interestingly, we would also clearly want $notModifiableList \in notModifiableList$.

The power-set concept, $\mathtt{Pow}(C)$, allows to capture in a natural way the interactions between concepts and metaconcepts. Considering again the example above, the statement "all the instances of species in the Red List are not allowed to be hunted" can be represented by the concept inclusion axiom: $RedListSpecies \sqsubseteq \mathtt{Pow}(CannotHunt)$, meaning that all the instances in the $RedListSpecies$ (as the class $Eagle$) are included in $CannotHunt$.

Motik has shown in [16] that the semantics of metamodeling adopted in OWL-Full leads to undecidability already for $\mathcal{ALC}$-Full, due to the free mixing of logical and metalogical symbols. In [16], limiting this free mixing but allowing atomic names to be interpreted as concepts and to occur as instances of other concepts, two alternative semantics (the Contextual π-semantics and the Hilog ν-semantics) are proposed for metamodeling. Decidability of $\mathcal{SHOIQ}$ extended with metamodeling is proved under either of the two proposed semantics.

Starting from [16], many other approaches to metamodeling have been proposed in the literature. Most of them [6, 11, 14, 9] are based on a Hilog semantics, while [19, 17] define extensions of OWL DL and of $\mathcal{SHIQ}$ (respectively), based on semantics interpreting concepts as well-founded sets.

Here, we propose an extension of $\mathcal{ALC}$ with power-set concepts and membership axioms among concepts, whose semantics is naturally defined using sets living in Ω-models (not necessarily well-founded). We prove that $\mathcal{ALC}^{\Omega}$ is decidable by defining, for any $\mathcal{ALC}^{\Omega}$ knowledge base K, a polynomial translation K^T into $\mathcal{ALCOI}$, exploiting the correspondence studied in [5] between the membership relation in the set theory and a normal modality. We show that the translation K^T enjoys the finite model property and exploit it in the proof of completeness of the translation. From the translation in $\mathcal{ALCOI}$ we also get an ExpTime upper bound on the complexity of satisfiability in $\mathcal{ALC}^{\Omega}$. Interestingly enough, our translation has strong relations with the first-order reductions in [8, 11, 14].

2 Preliminaries

2.1 The description logics $\mathcal{ALC}$ and $\mathcal{ALCOI}$

Let N_C be a set of concept names, N_R a set of role names and N_I a set of individual names. The set $\mathcal{C}$ of $\mathcal{ALC}$ *concepts* can be defined inductively as follows:

- $A \in N_C$, $\top$ and $\bot$ are concepts in $\mathcal{C}$;

- if $C, D \in \mathcal{C}$ and $R \in N_R$, then $C \sqcap D, C \sqcup D, \neg C, \forall R.C, \exists R.C$ are concepts in $\mathcal{C}$.

A knowledge base (KB) K is a pair $(\mathcal{T}, \mathcal{A})$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox. The TBox $\mathcal{T}$ is a set of concept inclusions (or subsumptions) $C \sqsubseteq D$, where C, D are concepts in $\mathcal{C}$. The ABox $\mathcal{A}$ is a set of assertions of the form $C(a)$ and $R(a,b)$ where C is a concept, $R \in N_R$, and $a,b \in N_I$.

An interpretation for $\mathcal{ALC}$ [2] is a pair $I = \langle \Delta, \cdot^I \rangle$ where Δ is a domain (a set whose elements are denoted by $x, y, z, \dots$) and $\cdot^I$ is an extension function that maps each concept name $C \in N_C$ to a set $C^I \subseteq \Delta$, each role name $R \in N_R$ to a binary relation $R^I \subseteq \Delta \times \Delta$, and each individual name $a \in N_I$ to an element $a^I \in \Delta$. The function $\cdot^I$ is extended to complex concepts as follows:

$$\top^I = \Delta \qquad \bot^I = \emptyset \qquad (\neg C)^I = \Delta \backslash C^I$$
$$(C \sqcap D)^I = C^I \cap D^I \qquad (C \sqcup D)^I = C^I \cup D^I$$
$$(\forall R.C)^I = \{x \in \Delta \mid \forall y.(x,y) \in R^I \rightarrow y \in C^I\}$$
$$(\exists R.C)^I = \{x \in \Delta \mid \exists y.(x,y) \in R^I \ \& \ y \in C^I\}$$

The notion of satisfiability of a KB in an interpretation is defined as follows:

Definition 1 (Satisfiability and entailment). *Given an $\mathcal{ALC}$ interpretation $I = \langle \Delta, \cdot^I \rangle$:*
- I satisfies an inclusion $C \sqsubseteq D$ if $C^I \subseteq D^I$;
- I satisfies an assertion $C(a)$ if $a^I \in C^I$;
- I satisfies an assertion $R(a,b)$ if $(a^I, b^I) \in R^I$.
Given a KB $K = (\mathcal{T}, \mathcal{A})$, an interpretation I satisfies $\mathcal{T}$ (resp. $\mathcal{A}$) if I satisfies all inclusions in $\mathcal{T}$ (resp. all assertions in $\mathcal{A}$); I is a model of K if I satisfies $\mathcal{T}$ and $\mathcal{A}$.
Let a query *F be either an inclusion $C \sqsubseteq D$ (where C and D are concepts) or an assertion $C(a)$. F* is entailed by *K, written $K \models F$, if for all models $I = \langle \Delta, \cdot^I \rangle$ of K, I satisfies F.*

Given a knowledge base K, the *subsumption* problem is the problem of deciding whether an inclusion $C \sqsubseteq D$ is entailed by K. The *instance checking* problem is the problem of deciding whether an assertion $C(a)$ is entailed by K. The *concept satisfiability* problem is the problem of deciding, for a concept C, whether C is consistent with K (i.e., whether there exists a model I of K, such that $C^I \neq \emptyset$).

In the following we will also consider the description logic $\mathcal{ALCOI}$ allowing inverse roles and nominals. For a role $R \in N_R$, its *inverse* is a role, denoted by R^-, which can be used in existential and universal restrictions with the following semantics: $(x,y) \in (R^-)^I$ if and only if $(y,x) \in R^I$. For a named individual $a \in N_I$, the *nominal* $\{a\}$ is the concept such that: $(\{a\})^I = \{a^I\}$.

2.2 The theory Ω

The first-order theory Ω consists of the following four axioms in the language with relational symbols $\in$ and $\subseteq$, and functional symbols $\cup$, $\backslash$, Pow:

$$x \in y \cup z \leftrightarrow x \in y \vee x \in z;$$
$$x \in y \backslash z \leftrightarrow x \in y \wedge x \notin z;$$
$$x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y);$$
$$x \in Pow(y) \leftrightarrow x \subseteq y.$$

In any Ω-model *everything* is supposed to be a set. Hence, a set will have (only) sets as its elements and circular definition of sets are not forbidden—i.e., for example, there are models of Ω in which there are sets admitting themselves as elements. Moreover, not postulating in Ω any *link* between membership $\in$ and equality—in axiomatic terms, having no *extensionality* (axiom)—, there exist Ω-models in which there are different sets with equal collection of elements.

The most natural Ω-model—in which different sets are, in fact, always extensionally different—is the collection of well-founded sets $\mathsf{HF} = \mathsf{HF}^0 = \bigcup_{n\in\mathbb{N}} \mathsf{HF}_n$, where:

$$\mathsf{HF}_0 = \emptyset; \qquad \mathsf{HF}_{n+1} = Pow(\mathsf{HF}_n).$$

In HF^0 every system of set-theoretic equations of the form:

$$\begin{cases} x_1 = \{x_{1,1}, \dots, x_{1,m_1}\}; \\ x_2 = \{x_{2,1}, \dots, x_{2,m_2}\}; \\ \vdots \qquad\quad \vdots \\ x_n = \{x_{n,1}, \dots, x_{n,m_n}\}, \end{cases}$$

where $x_{i,j}$ for $j = 1, \dots, m_i$ is one among $x_1, \dots x_{i-1}$, finds a unique solution.

If we drop the index-ordering restriction on variables appearing in the right-hand-side of set-theoretic equations (thereby allowing equations such as $x = \{x\}$), in order to guarantee the existence of solutions in the model we need to work with universes *larger* than HF. The most natural (and minimal) among them is a close relative of HF^0 and goes under the name of $\mathsf{HF}^{1/2}$ (see [1, 18]).

Finally, a further enrichment of both HF^0 and $\mathsf{HF}^{1/2}$ is obtained by adding *atoms*, that is copies of the empty-set, to be denoted by $\mathbf{a}_1, \mathbf{a}_2, \dots$ and collectively represented by $\mathbb{A} = \{\mathbf{a}_1, \mathbf{a}_2, \dots\}$. The resulting universes will be denoted by $\mathsf{HF}^0(\mathbb{A})$ and $\mathsf{HF}^{1/2}(\mathbb{A})$.

A complete discussion relative to universes of sets to be used as models of Ω goes beyond the scope of this paper. However, it is convenient to point out that, in all cases of interest for us here, an especially simple view of Ω-models can be given using *finite graphs*. Actually, HF^0 or $\mathsf{HF}^{1/2}$ can be seen as the collection of finite graphs (either acyclic or cyclic, respectively), where sets are represented by nodes and arcs depict the membership relation among sets (see [18]). Given one such *membership* graph G it is convenient to single out a special node (the *point* of the graph), to isolate the specific set for which the description is introduced.

In the next section, we will regard the domain Δ of a DL interpretation as a fragment of the universe of an Ω-model, i.e. Δ will be regarded as a set of sets of the theory Ω rather than as a set of individuals, as customary in description logics.

3 The description logic $\mathcal{ALC}^{\Omega}$

We start from the observation that in $\mathcal{ALC}$ concepts are interpreted as sets (namely, sets of domain elements) and we generalize $\mathcal{ALC}$ by allowing concepts to be interpreted as sets of the set theory Ω. In addition, we extend the language of $\mathcal{ALC}$ by introducing the power-set as a new concept constructor, and allowing membership relations among concepts in the knowledge base. We call $\mathcal{ALC}^{\Omega}$ the resulting extension of $\mathcal{ALC}$.

As before, let N_I , N_C, and N_R be the set of individual names, concept names, and role names in the language, respectively. In building complex concepts, in addition to the constructs of $\mathcal{ALC}$, we also consider the difference $\backslash$ and the power-set $\texttt{Pow}$ constructs. The set of $\mathcal{ALC}^{\Omega}$ *concepts* are defined inductively as follows:

- $A \in N_C$, $\top$ and $\bot$ are $\mathcal{ALC}^{\Omega}$ *concepts*;
- if C, D are $\mathcal{ALC}^{\Omega}$ concepts and $R \in N_R$, then the following are $\mathcal{ALC}^{\Omega}$ *concepts*:

$$C \sqcap D, C \sqcup D, \neg C, C \backslash D, \texttt{Pow}(C), \forall R.C, \exists R.C$$

While the concept $C \backslash D$ can be easily defined as $C \sqcap \neg D$ in $\mathcal{ALC}$, this is not the case for the concept $\texttt{Pow}(C)$. Informally, the instances of concept $\texttt{Pow}(C)$ are all the subsets of the instances of concept C visible in the domain Δ (see below).

Besides ABox assertions of the form $C(a)$ with $a \in N_I$, we allow in the ABox *concept membership axioms* and *role membership axioms*, respectively, of the form: $C \in D$ and $(C, D) \in R$, where C and D are $\mathcal{ALC}^{\Omega}$ concepts and R is a role name.

Considering again the example from the Introduction, the additional expressivity of the language, in which general concepts (and not only concept names) can be instances of other concepts, allows for instance to represent the fact that polar bears are in the red list of endangered species, by the axiom $\mathit{Polar} \sqcap \mathit{Bear} \in \mathit{RedListSpecies}$. We can further represent the fact the polar bears are more endangered than eagles by adding a role $\mathit{moreEndangered}$ and the role membership axiom $(\mathit{Polar} \sqcap \mathit{Bear}, \mathit{Eagle}) \in \mathit{moreEndangered}$. Observe that, as shown in [16], the meaning of the sentence $\mathit{RedListSpecies} \sqsubseteq \texttt{Pow}(\mathit{CannotHunt})$ (i.e. "all the instances of species in the Red List are not allowed to be hunted"), could be captured by combining the ν-semantics with the Semantic Web Rule Language (SWRL) [12], but not by the ν-semantics alone.

We define a semantics for $\mathcal{ALC}^{\Omega}$ by extending the $\mathcal{ALC}$ semantics in Section 2.1 to capture the meaning of concepts (including concept $\texttt{Pow}(C)$) as elements (sets) of the domain Δ, chosen as a *transitive* set (i.e. a set x satisfying $(\forall y \in x)(y \subseteq x)$) in a model of Ω. Roles are interpreted as binary relations over the domain Δ. Individual names are interpreted as elements of a set of atoms $\mathbb{A}$ from which the sets in Δ are built.

Definition 2. *An interpretation for $\mathcal{ALC}^{\Omega}$ is a pair $I = \langle \Delta, \cdot^I \rangle$ over a set of atoms $\mathbb{A}$ where:*

- *the non-empty domain Δ is a transitive set chosen in a model $\mathcal{M}$ of Ω over the atoms in $\mathbb{A}$ (we let $\mathcal{U}$ be the universe of the model $\mathcal{M}$);*[2]
- *the extension function $\cdot^I$ maps each concept name $A \in N_C$ to an element $A^I \in \Delta$; each role name $R \in N_R$ to a binary relation $R^I \subseteq \Delta \times \Delta$; and each individual name $a \in N_I$ to an element $a^I \in \mathbb{A} \subseteq \Delta$.*

The function $\cdot^I$ is extended to complex concepts of $\mathcal{ALC}^{\Omega}$, as in Section 2.1 for $\mathcal{ALC}$, but for the two additional cases: $(\texttt{Pow}(C))^I = \mathit{Pow}(C^I) \cap \Delta$ and $(C \backslash D)^I = (C^I \backslash D^I)$.

Observe that $\mathbb{A} \subseteq \Delta \in \mathcal{U}$. As Δ is not guaranteed to be closed under union, intersection, etc., the interpretation C^I of a concept C is a set in $\mathcal{U}$ but not necessarily an element

[2] In the following, for readability, we will denote by $\in$, Pow, $\cup$, $\backslash$ (rather than $\mathit{Pow}^{\mathcal{M}}$, $\cup^{\mathcal{M}}$, $\backslash^{\mathcal{M}}$) the interpretation in a model $\mathcal{M}$ of the predicate and function symbols $\in$, Pow, $\cup$, $\backslash$.

of Δ. However, given the interpretation of the power-set concept as the portion of the (set-theoretic) power-set *visible in* Δ, it easy to see by induction that, for each C, the extension of C^I is a subset of Δ.

Given an interpretation I, the satisfiability of inclusions and assertions is defined as in $\mathcal{ALC}$ interpretations (Definition 1). Satisfiability of (concept and role) membership axioms in an interpretation I is defined as follows: I *satisfies* $C \in D$ if $C^I \in D^I$; I *satisfies* $(C, D) \in R$ if $(C^I, D^I) \in R^I$. With this addition, the notions of satisfiability of a KB and of entailment in $\mathcal{ALC}^\Omega$ (denoted $\models_{\mathcal{ALC}^\Omega}$) can be defined as in Section 2.1.

The problem of instance checking in $\mathcal{ALC}^\Omega$ includes both the problem of verifying whether an assertion $C(a)$ is a logical consequence of the KB and the problem of verifying whether a membership $C \in D$ is a logical consequence of the KB (i.e., whether C is an instance of D).

In the next section, we define a polynomial encoding of the language $\mathcal{ALC}^\Omega$ into the description logic $\mathcal{ALCOI}$.

4 Translation of $\mathcal{ALC}^\Omega$ into $\mathcal{ALCOI}$

To provide a proof method for $\mathcal{ALC}^\Omega$, we define a translation of $\mathcal{ALC}^\Omega$ into the description logic $\mathcal{ALCOI}$, including inverse roles and nominals. In [5] the membership relation $\in$ is used to represent a normal modality R of a modal logic. In this section, vice-versa, we exploit the correspondence between $\in$ and the accessibility relation of a modality by introducing a new (reserved) role e in N_R to represent the inverse of the membership relation: in any interpretation I, $(x, y) \in e^I$ will stand for $y \in x$. The idea underlying the translation is that each element u of the domain Δ in an $\mathcal{ALCOI}$ interpretation $I = \langle \Delta, \cdot^I \rangle$ can be regarded as the set of all the elements v such that $(u, v) \in e^I$.

The translation of an $\mathcal{ALC}^\Omega$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ into $\mathcal{ALCOI}$ can be defined as follows. First, we associate each concept C of $\mathcal{ALC}^\Omega$ to a concept C^T of $\mathcal{ALCOI}$ by replacing all occurrences of the power-set constructor `Pow` with a concept involving the universal restriction $\forall e$ (see below). More formally, we (inductively) define the translation C^T of C by simply recursively replacing every subconcept $\texttt{Pow}(D)$ appearing in C by $\forall e.D^T$, while the translation T commutes with concept constructors in all other cases.

Semantically this will result in interpreting any (sub)concept $(\texttt{Pow}(D))^I$ by

$$(\forall e.D)^I = \{x \in \Delta \mid \forall y((x, y) \in e^I \rightarrow y \in D^I)\},$$

which, recalling that $(x, y) \in e^I$ stands for $y \in x$, will characterize the collection of subsets of D^I *visible in* Δ (i.e. subsets of D^I which are also elments of Δ): $(\forall e.D)^I = \{x \in \Delta \mid \forall y(y \in x \rightarrow y \in D^I)\}$, that is, $(\forall e.D)^I = \{x \in \Delta \mid x \subseteq D^I)\} = \texttt{Pow}(D^I) \cap \Delta = (\texttt{Pow}(D))^I$, as expected.

4.1 Translating TBox, ABox, and queries

We define a new TBox, $\mathcal{T}^T$, by introducing, for each inclusion $C \sqsubseteq D$ in $\mathcal{T}$, the inclusion $C^T \sqsubseteq D^T$ in $\mathcal{T}^T$. Additionally, for each (complex) concept C occurring in the knowledge base K (or in the query) on the l.h.s. of a membership axiom $C \in D$

or $(C, D) \in R$, we extend N_I with *a new individual name*[3] e_C and we add the concept equivalence:

$$C^T \equiv \exists e^-.\{e_C\}. \tag{1}$$

in $\mathcal{T}^T$. From now on, new individual names such as e_C will be called *concept individual names*. This equivalence is intended to capture the property that, in all the models $I = \langle \Delta, \cdot^I \rangle$ of K^T, e_C^I is in relation e^I with all and only the instances of concept C^T, i.e., for all $y \in \Delta$, $(e_C^I, y) \in e^I$ if and only if $y \in (C^T)^I$.

As in the case of the power-set constructor, this fact can be verified by analyzing the semantics of $\exists e^-.\{e_C\}$:

$$(\exists e^-.\{e_C\})^I = \{x \in \Delta \mid \exists y((x,y) \in (e^-)^I \wedge y \in (\{e_C\})^I\},$$

which, recalling that e stands for $\ni$ and interpreting the nominal, will stand for

$$(\exists e^-.\{e_C\})^I = \{x \in \Delta \mid \exists y(x \in y \wedge y \in \{e_C^I\}\} = \{x \in \Delta \mid x \in e_C^I\},$$

which, by the concept equivalence $C^T \equiv \exists e^-.\{e_C\}$, is as to say that e_C^I and $(C^T)^I$ have the same extension.

Remark 1. It is important to notice that every concept individual name of the sort e_C introduced above—that is, every individual name whose purpose is that of providing a name to the extension of C^I—, in general turns out to be in relation e with other elements of the domain Δ of I (unless C is an inconsistent concept and its extension is empty). This is in contrast with the assumption relative to other "standard" individual names $a \in N_I$, for which we will require $(\neg\exists e.\top)(a)$ (see below).

We define $\mathcal{A}^T$ as the set of assertions containing:

- for each concept membership axiom $C \in D$ in $\mathcal{A}$, the assertion $D^T(e_C)$,
- for each role membership axiom $(C, D) \in R$ in $\mathcal{A}$, the assertion $R(e_C, e_D)$,
- for each assertion $D(a)$ in $\mathcal{A}$, the assertion $D^T(a)$,
- for each assertion $R(a, b)$ in $\mathcal{A}$, the assertion $R(a, b)$ and, finally,
- for each (standard) individual name $a \in N_I$, the assertion $(\neg\exists e.\top)(a)$.

As noticed above, the last requirement forces all named individuals (in the language of the initial knowledge base K) to be interpreted as domain elements which are not in relation e with any other element.

Let $K^T = (\mathcal{T}^T, \mathcal{A}^T)$ be the knowledge base obtained by translating K into $\mathcal{ALCOI}$.

Example 1. Let $K = (\mathcal{T}, \mathcal{A})$ be the knowledge base considered above:
$\mathcal{T} = \{\mathit{RedListSpecies} \sqsubseteq \mathtt{Pow}(\mathit{CannotHunt})\}$ and
$\mathcal{A} = \{\mathit{Eagle}(\mathit{harry}), \mathit{Eagle} \in \mathit{RedListSpecies},\ \mathit{Polar} \sqcap \mathit{Bear} \in \mathit{RedListSpecies}\}$.
By the translation above, we obtain:
$\mathcal{T}^T = \{\mathit{RedListSpecies} \sqsubseteq \forall e.\mathit{CannotHunt},$
$\quad \mathit{Eagle} \equiv \exists e^-.\{e_{\mathit{Eagle}}\},\ \mathit{Polar} \sqcap \mathit{Bear} \equiv \exists e^-.\{e_{\mathit{Polar}\sqcap\mathit{Bear}}\}\ \}$
$\mathcal{A}^T = \{\mathit{Eagle}(\mathit{harry}), \mathit{RedListSpecies}(e_{\mathit{Eagle}}), \mathit{RedListSpecies}(e_{\mathit{Polar}\sqcap\mathit{Bear}}),$
$\quad (\neg\exists e.\top)(\mathit{harry})\ \}$

[3] The symbol e_C should remind the *e*-xtension of C.

K^T entails $CannotHunt(harry)$ in $\mathcal{ALCOI}$. In fact, from $RedListSpecies(e_{Eagle})$ and $RedListSpecies \sqsubseteq \forall e.CannotHunt$, it follows that, in all models of K^T, $e^I_{Eagle} \in (\forall e.CannotHunt)^I$. Furthermore, from $Eagle \equiv \exists e^-.\{e_{Eagle}\}$ and the assertion $Eagle(harry)$, it follows that $(e^I_{Eagle}, harry^I) \in e^I$ holds. Hence, $harry^I \in Cannot Hunt^I$. As this holds in all models of K^T, $CannotHunt(harry)$ is a logical consequence of K^T. It is easy to see that $Eagle \sqsubseteq CannotHunt$ follows from K^T as well.

Let F be a *query* of the form $C \sqsubseteq D$, $C(a)$ or $C \in D$ We assume that all the individual names, concept names and role names occurring in F also occur in K and we define a translation F^T of the query F as follows:

- if F is a subsumption $C \sqsubseteq D$, then F^T is the subsumption $C^T \sqsubseteq D^T$;
- if F is an assertion $C(a)$, then F^T is the assertion $C^T(a)$;
- if F is a membership axiom $C \in D$ (respectively, $(C, D) \in R$), then F^T is the assertion $D^T(e_C)$ (respectively, $R(e_C, e_D)$).

In the following we state the soundness and completeness of the translation of an $\mathcal{ALC}^\Omega$ knowledge base into $\mathcal{ALCOI}$.

Proposition 1 (Soundness of the translation). *The translation of an $\mathcal{ALC}^\Omega$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ into $\mathcal{ALCOI}$ is sound, that is, for any query F:*

$$K^T \models_{\mathcal{ALCOI}} F^T \Rightarrow K \models_{\mathcal{ALC}^\Omega} F.$$

For the sake of brevity a proof of the above proposition, which is given along standard lines, is not given here.

Before proving the completeness of the translation of $\mathcal{ALC}^\Omega$ into $\mathcal{ALCOI}$, we show that, if the translation K^T of a knowledge base K in $\mathcal{ALC}^\Omega$ has a model in $\mathcal{ALCOI}$, then it also has a finite model.

Proposition 2. *Let K be a knowledge base in $\mathcal{ALC}^\Omega$ and let K^T be its translation in $\mathcal{ALCOI}$. If K^T has a model in $\mathcal{ALCOI}$, then it has a finite model.*

Proof. We prove this result by providing an alternative (but equivalent) translation $K^{T(\neg)}$ of K in the description logic $\mathcal{ALC}(\neg)$, using a single negated role $\neg e$.

$\mathcal{ALC}(\neg)$ extends $\mathcal{ALC}$ with role complement operator, where, for any role R, the role $\neg R$ is *the negation of role* R, where $(x, y) \in (\neg R)^I$ if and only if $(x, y) \notin R^I$. In the translation, we exploit $\neg e$ to capture non-membership, where $(x, y) \in (\neg e)^I$ if and only if $(x, y) \notin e^I$ (i.e., in set terms, $y \notin x$). Decidability of concept satisfiability in $\mathcal{ALC}(\neg)$ has been proved by Lutz and Sattler in [15]. The finite model property of a language with a single negated role $\neg e$ can be proved as done in [7] (Section 2) for a logic with the "window modality", by standard filtration, extended to deal with additional K-modalities (for the other roles) as in the proof in [3]. Indeed, as observed in [15], the "window operator" $\boxdot$ studied in [7] is strongly related to a negated modality, as $\boxdot\, \phi$ can be written as $[\neg R]\neg\phi$.

The translation $K^{T(\neg)}$ can be defined modifying K^T by replacing the concept equivalence $C^T \equiv \exists e^-.\{e_C\}$ with the assertions: $(\forall e.C^T)(e_C)$ and $(\forall(\neg e).(\neg C^T))(e_C)$.

One can show that any model $I = (\Delta, \cdot^I)$ of $K^{T(\neg)}$ is a model of K^T in $\mathcal{ALCOI}$, and vice-versa (considering the usual interpretation of negated roles, inverse roles and nominals). In fact, the semantic meaning of the assertion $(\forall e.C^T)(e_C)$ is the following:

for all $x \in \Delta$, $(e_C^I, x) \in e^I \Rightarrow x \in (C^T)^I$,

which is equivalent to the meaning of $\exists e^-.\{e_C\} \sqsubseteq C^T$.

The semantic meaning of the assertion $(\forall(\neg e).(\neg C^T))(e_C)$ is: for all $x \in \Delta$, $(e_C^I, x) \notin e^I \Rightarrow x \notin (C^T)^I$, i.e., for all $x \in \Delta$, $x \in (C^T)^I \Rightarrow (e_C^I, x) \in e^I$, which is the semantic meaning of $C^T \sqsubseteq \exists e^-.\{e_C\}$.

We conclude the proof by observing that, if K^T has a model, it is a model of $K^{T(\neg)}$. Then, by the finite model property, $K^{T(\neg)}$ must have a finite model which is, in turn, a finite model of K^T. □

To conclude our analysis we now prove the completeness of our translation.

Proposition 3 (Completeness of the translation). *The translation of an $\mathcal{ALC}^\Omega$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ into $\mathcal{ALCOI}$ is complete, that is, for any query F:*

$$K \models_{\mathcal{ALC}^\Omega} F \Rightarrow K^T \models_{\mathcal{ALCOI}} F^T.$$

Proof (sketch). The proof is by contraposition. Assume that $K^T \not\models_{\mathcal{ALCOI}} F^T$. Then there is a model $I = \langle \Delta, \cdot^I \rangle$ of K^T in $\mathcal{ALCOI}$ such that I falsifies F.

We show that we can build a model $J = \langle \Lambda, \cdot^J \rangle$ of K in $\mathcal{ALC}^\Omega$, where the domain Λ is a transitive set in the universe $\mathsf{HF}^{1/2}(\mathbb{A})$ consisting of all the hereditarily finite rational hypersets built from atoms in $\mathbb{A} = \{\mathbf{a_0}, \mathbf{a_1}, \ldots\}$.

We define Λ starting from the graph[4] $G = \langle \Delta, e^I \rangle$, whose nodes are the elements of Δ and whose arcs are the pairs $(x, y) \in e^I$. Notice that, by Proposition 2, the graph G can be assumed to be finite. Intuitively, an arc from x to y in G stands for the fact that $y \in x$.

At this point, let $\Delta_0 = \{d_1, \ldots, d_m\}$ be the elements of Δ which, in the model $I = \langle \Delta, \cdot^I \rangle$, are not in relation e^I with any other element in Δ and are non equal to the interpretation of any concept individual name e_C (that is, $d_j \in \Delta_0$ iff there is no y such that $(d_j, y) \in e^I$ and there is no concept C such that $d_j = e_C^I$). For any given $d \in \Delta$ we define the following hyperset $M(d)$:

$$M(d) = \begin{cases} \mathbf{a_k} & \text{if } d = d_k \in \Delta_0, \\ \{M(d') \mid (d, d') \in e^I\} & \text{otherwise.} \end{cases} \quad (2)$$

Observe that, for the concepts C occurring on the l.h.s. of membership axioms, as axiom $C^T \equiv \exists e^-.\{e_C\}$ is satisfied in the model I of K^T, it holds that $d' \in (C^T)^I$ iff $(e_C^I, d') \in e^I$. Therefore, for $d = e_C^I$, $M(d) = M(e_C^I) = \{M(d') \mid (e_C^I, d') \in e^I\}$ $= \{M(d') \mid d' \in (C^T)^I\}$.

The above definition uniquely determines hypersets in $\mathsf{HF}^{1/2}(\mathbb{A})$. This follows from the fact that all finite systems of (finite) set-theoretic equations have a solution in $\mathsf{HF}^{1/2}(\mathbb{A})$. As a matter of fact, whenever the graph G is acyclic, the definition of $M(d)$ identifies a standard (recursively given) hereditarily finite set[5].

[4] Strictly speaking the graph G introduced here is not really necessary: it is just mentioned to single out the membership relation $\in$ from e^I more clearly.

[5] More generally, when e^I is a well-founded relation, $M(\cdot)$ is a set-theoretic "rendering" of e^I: the so-called *Mostowski collapse* of e^I (see [13]).

Our task now is to complete the definition of $J = \langle \Lambda, \cdot^J \rangle$ in such a way to prove that J is a model of K in $\mathcal{ALC}^\Omega$ falsifying F. The definition is completed as follows:

- $\Lambda = \{M(d) \mid d \in \Delta\}$;
- for all $B \in N_C$, $B^J = \{M(d) \mid d \in B^I\}$;
- for all roles $R \in N_R$ such that $R \neq e$, $R^J = \{(M(d), M(d')) \mid (d, d') \in R^I\}$;
- for all standard name individuals $a \in N_I$ such that $a^I = d_k$, let $a^J = M(d_k) = \mathbf{a_k} \in \mathbb{A}$.

To complete the proof it can be shown that, for all $d \in \Delta$, $M(d) \in C^J$ if and only if $d \in (C^T)^I$, which is used to show that J is a model of K that falsifies F. □

As the translation of $\mathcal{ALC}^\Omega$ into $\mathcal{ALCOI}$ is polynomial (actually, linear) in the size of the knowledge base (and of the query) the following complexity result follows.

Proposition 4. *Concept satisfiability in $\mathcal{ALC}^\Omega$ is an* EXPTIME*-complete problem.*

The hardness comes from the EXPTIME-hardness of $\mathcal{ALC}$ concept satisfiability w.r.t. a set of inclusions [21]; the upper bound from the EXPTIME upper bound for $\mathcal{SHOI}$ [10].

5 Conclusions and related work

In this paper we have shown that the similarities between description logics and set theory can be exploited to introduce in DLs the new power-set construct and to allow for (possibly circular) membership relationships among arbitrary concepts. We have defined the description logic $\mathcal{ALC}^\Omega$, combining $\mathcal{ALC}$ with the set theory Ω, and defined its semantics whose interpretation domains are fragments of the domains of Ω-models. $\mathcal{ALC}^\Omega$ allows membership axioms among concepts as well as the power-set construct which, up to our knowledge, has not been considered for description logics before. We have shown that an $\mathcal{ALC}^\Omega$ knowledge base can be polynomially translated into an $\mathcal{ALCOI}$ knowledge base. Soundness and completeness of the translation provide, besides decidability, an EXPTIME upper bound for satisfiability in $\mathcal{ALC}^\Omega$.

The power-set construct allows to capture in a very natural way the interactions between concepts and metaconcepts, adding to the language of $\mathcal{ALC}$ the expressivity of metamodelling. The issue of metamodelling has been analysed by Motik in [16], proving that metamodelling in $\mathcal{ALC}$-Full is already undecidable due to free mixing of logical and metalogical symbols. Two decidable semantics, a contextual π semantics and a Hilog ν-semantics, are introduced in [16] for a language extending $\mathcal{SHOIQ}$ with metamodelling, where concept names, role names and individual names are not disjoint. This possibility of using the same name in different contexts is introduced in OWL 1.1 and then in OWL 2 through *punning*[6]. As a difference, in this paper, we consider concept names, role names and individual names to be disjoint, we allow concepts (and not only concept names) to be instances of other concepts, by membership axioms, while we do not allow role names as instances.

As in [16], DeGiacomo et al. [6] and Homola et al. [11] employ an Hilog-style semantics to define $Hi(\mathcal{SHIQ})$ and $\mathcal{TH}(\mathcal{SROIQ})$, respectively. While [16] and [6]

[6] https://www.w3.org/2007/OWL/wiki/Punning

define untyped higher-order languages which, as $\mathcal{ALC}^{\Omega}$, allow a concept to be an instance of itself, [11] defines a typed higher-order extension of $\mathcal{SROIQ}$ allowing for a hierarchy of concepts, where concept names of order t can only occur as instances of concepts of order $t + 1$. In $\mathcal{TH}(\mathcal{SROIQ})$ [11] there is a strict separation between concepts and roles (as in $\mathcal{ALC}^{\Omega}$) and decidability is proved by a polynomial first-order reduction into $\mathcal{SROIQ}$, which generalizes the reduction in [8] to an arbitrary number of orders. The translation in [11] introduces axioms $A' \equiv \exists instanceOf.\{c_{A'}\}$, for each atomic concept A', axioms which are quite similar to our axiom (1), that we need for the concepts C occurring in the knowledge base on the left hand side of membership axioms.

In $Hi(\mathcal{SHIQ})$ [6], complex concept and role expressions can occur as instances of other concepts as in $\mathcal{ALC}^{\Omega}$. A polynomial translation of $Hi(\mathcal{SHIQ})$ into $\mathcal{SHIQ}$ is defined and a study of the complexity of higher-order query answering is provided.

Kubincova et al. in [14] propose a Hylog-style semantics by dropping the ordering requirement in [11] and allowing the $instanceOf$ role, with a fixed interpretation, to be used in axioms as any other role. The interpretation of role $instanceOf$ does not correspond exactly to the interpretation of e^{-} in our translation, as we do not introduce axiom (1) for all the concept names in N_C, while we introduce it for all the (possibly complex) concepts occurring as instances in some membership axiom.

Pan and Horrocks in [19] and Motz et al. in [17] define extensions of OWL DL and of $\mathcal{SHIQ}$ (respectively), based on semantics interpreting concepts as well-founded sets. In particular, [17] adds to $\mathcal{SHIQ}$ meta-modelling axioms equating individuals to concepts, without requiring that the instances of a concept need to stay in the same layer, and develop a tableau algorithm as an extension of the one for $\mathcal{SHIQ}$.

In [9] Gu introduces the language Hi(Horn-SROIQ), an extension of Horn-SROIQ which allows classes and roles to be used as individuals based on the ν-semantics [16]. ν-satisfiability and conjunctive query answering are shown to be reducible to the corresponding problems in Horn-SROIQ.

A set-theoretic approach in DLs has been adopted by Cantone et al. in [4] for determining the decidability of higher order conjunctive query answering in the description logic $\mathcal{DL}_D^{4,\times}$ (where concept and role variables may occur in queries), as well as for developing a tableau based procedure for calculating the answer sets from a $\mathcal{DL}_D^{4,\times}$ knowledge base, thus providing means for dealing with several well-known ABox reasoning tasks.

We expect that the approach of extending $\mathcal{ALC}$ with Ω can be adopted as well for more expressive DLs, which do not enjoy the finite model property. However, when the finite model property does not hold, there may be models of the translated knowledge base K^T containing domain elements being in the relation e with infinitely many elements, and corresponding to infinite sets. The completeness proof of Proposition 3 does not apply to this case and we leave the study of this case for future investigation.

Other possible directions for future investigation concern: the treatment of roles as individuals, which has not been considered as an option in $\mathcal{ALC}^{\Omega}$; restricting the semantics to well-founded sets to avoid circular definitions of sets; translating $\mathcal{ALC}^{\Omega}$ into the set theory Ω, which may open to the possibility of exploiting proof methods developed for set theories in reasoning with DLs, as an alternative to translating to DLs.

Acknowledgement: This research is partially supported by INDAM-GNCS Project 2018 “Metodi di prova orientati al ragionamento automatico per logiche non-classiche”.

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