An Algorithm to Construct Generalized Voronoi Diagrams with Fuzzy Parameters Based on the Theory of Optimal Partitioning and Neuro-Fuzzy Technologies

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Abstract. An algorithm to construct a generalized Voronoi diagram (VD) in the presence of fuzzy parameters is proposed with optimal location of a finite number of generator points in a bounded set of n-dimensional Euclidean space E_n. The algorithm is based on the formulation of the corresponding continuous problem of optimal partitioning of a set (OPS) from E_n into non-intersecting subsets with a partitioning quality criterion providing the corresponding VD form with fuzzy parameters. The algorithm was developed using a synthesis of methods for solving OPS theory problems, neuro-fuzzy technologies and modifications of the N.Z.Shor's r-algorithm for solving non-smooth optimization problems.

Keywords: generalized Voronoi diagram, problem of optimal partitioning of sets, optimal location of generator points, neuro-fuzzy technologies, N.Z.Shor's r-algorithm, non-smooth optimization problems.

1 Introduction

As is known [1, 2], a Voronoi diagram (VD) of a given finite set $M = \{\tau_1, \tau_2, ..., \tau_N\}$ of points in a plane (space), which are called generator points, is a mathematical object representing such a partitioning of a plane (space) at which each domain (Voronoi cell) of this partitioning forms a set of elements located closer to one of the points of set M than to any other point of this set.

Voronoi diagrams in two- and three-dimensional spaces are used in various fields of applied sciences [3, 4]. Currently, there are several hundreds of references on VD and their applications in various fields. Substantive reviews of some of them are given in [3-5]. There one can also find a general review of recent technical results and applications with links to primary sources. Despite the fact that many well-known algorithms for VD constructing cost $O(N \log N)$ time, and some of these algorithms even use $O(N)$ time, all of them are very complex. As for the VD for the case of a space of arbitrary dimension, its construction is associated with significant algorithmic problems. Indeed, it is known [2] that for a given number N of generator points, the number of elements needed to describe VD increases exponentially depending on the space dimension.

This paper is devoted to describing an algorithm to construct a generalized VD in the presence of fuzzy parameters with optimal placement of a finite number N of generator points in a bounded set Ω of n-dimensional Euclidean space E_n $(n \geq 2)$. The algorithm is developed on the basis of a synthesis of methods for solving OSP theory problems (see [3-5]) with neuro-fuzzy technologies (methodologies) (see [6-10]) and modifications of the N.Z.Shor's r-algorithm for solving non-smooth optimization problems [11, 12]. This algorithm is based on:

– formulating the corresponding problem of optimal partitioning of a set (OPS) of E_n into non-intersecting subsets with a partitioning quality criterion providing the corresponding VD form with fuzzy parameters;

– applying the mathematical and algorithmic apparatus described in [3] to solve such a problem, which includes the N.Z.Shor's r-algorithm [11, 12] as a part;

– using a neuro-fuzzy technology [6-10].

The basis of the above mathematical and algorithmic apparatus is the following general idea [3-5]. The initial OSP problems that are mathematically formulated as infinite-dimensional optimization problems, are reduced through the Lagrange functional to auxiliary finite-dimensional non-smooth maximization problems or non-smooth max-min problems, for the numerical solution of which modern efficient methods of non-differentiable optimization are used, namely, various modifications of the Shor's r-algorithm. The peculiarity of this approach is the fact that the solution of the initial infinite-dimensional optimization problems can be obtained analytically in an explicit form, and the analytic expression can include parameters that are sought as an optimal solution of the above-mentioned auxiliary finite-dimensional optimization problems with non-smooth target functions.

2 Review of the Literature

The algorithms to construct the standard VD with fuzzy parameters and its various generalizations that are proposed in [3-5] and in cited there sources have some advantages compared with those known in scientific literature [2, 13-20]:

− do not depend on the dimension of space E_n containing the partitionable bounded set into subsets (the problem is only reduced to the calculation of multidimensional integrals);

− do not depend on the geometry of the partitionable set;

− due to their high calculation speed, they are applicable for large dimensional problems (100, 200, 300 or more generator points);

− are applicable not only for the Euclidean metric, but also for the metrics of Chebyshev, Manhattan and others;

− the complexity of VD constructing algorithms based on the described approach does not increase while increasing numbers of generator points;

− are applicable to the construct not only VD with given number of generator points with their fixed location, but also with the optimal location of these points in a bounded set of space E_n;

− the result of this universal approach is the ability to easily construct not only the already known VD, but also new ones.

The universality of proposed in [3-5] approach to construct VD is confirmed by that models and methods to solve OSP problems can be generalized to the case of fuzzy specification of initial parameters of the problem or to the case of requirement of a fuzzy partitioning of the set, where VD also may be fuzzy.

We first give some information from [3-5] on constructing generalized VD with clear parameter on the basis of the above-mentioned mathematical apparatus.

As is known [2, 4], ***the standard (classical) Voronoi diagram*** of a finite set $M = \{\tau_1, \tau_2, ..., \tau_N\} \subset E_n$ of generator points $\tau_i = (\tau_i^{(1)}, \tau_i^{(2)}, ..., \tau_i^{(n)})$, $i = \overline{1, N}$, in n-dimensional Euclidean space E_n $(n \ge 2)$ is the set of Voronoi polytopes:

$$Vor(\tau_i) = \left\{ x \in \mathrm{E}_n : \ c(x, \tau_i) \le c(x, \tau_j),\ j = \overline{1, N},\ j \ne i \right\},\ i = \overline{1, N}, \qquad (1)$$

predefined points $\tau_1, ..., \tau_N$, where $c(x, y)$ is a metric in E_n, which may be defined as Euclidean, Manhattan and Chebyshev one [4]. In other words, VD of a finite set $M = \{\tau_1, \tau_2, ..., \tau_N\}$ from E_n $(n \ge 2)$ is the following set:

$$Vor(\mathrm{M}) = \bigcup_{\tau_i \in \mathrm{M}} Vor(\tau_i), \qquad (2)$$

where $mes\big(Vor(\tau_i) \bigcap Vor(\tau_j)\big) = 0$, $i, j = \overline{1, N}$ $(i \ne j)$; $mes(\cdot)$ is a Lebesgue measure. E_n space partitioning into Voronoi polytopes $Vor(\tau_i)$, $i = \overline{1, N}$ of given set $M = \{\tau_1, \tau_2, ..., \tau_N\}$ is monosemantic.

A brief review of some generalizations of the standard VD available in the scientific literature is given in book [4], for example:

– Additively Weighted VD of a set $M = \{\tau_1, \tau_2, ..., \tau_N\} \subset E_n$:

$$AW\ Vor(\mathrm{M}) = \bigcup_{\tau_i \in \mathrm{M}} AW\ Vor(\tau_i),$$

$$AW\ Vor(\tau_i) = \left\{ x \in \mathrm{E}_n : \ c(x, \tau_i) - w_i \le c(x, \tau_j) - w_j,\ j = \overline{1, N},\ j \ne i \right\}, \quad i = \overline{1, N},$$

where each generator point $\tau_i \in \mathrm{M}$ has a weight $w_i > 0$ $(i = \overline{1, N})$ added to the function specifying the distance while measurement;

– Multiplicatively Weighted VD of a set $M = \{\tau_1, \tau_2, ..., \tau_N\} \subset E_n$:

$$MW\ Vor(\mathrm{M}) = \bigcup_{\tau_i \in \mathrm{M}} MW\ Vor(\tau_i),$$

where each Voronoi polytopes

$$MW\ Vor(\tau_i) = \left\{ x \in \mathrm{E}_n : \ c(x, \tau_i) / w_i \le c(x, \tau_j) / w_j,\ j = \overline{1, N},\ j \ne i \right\},$$

$i = \overline{1, N}$, is a set of points in space, the weighted distance from which to the generator point $\tau_i \in \mathrm{M}$ does not exceed the weighted distance to any other point of M ($w_i > 0$, $i = \overline{1, N}$ as before are given weights).

Some other generalizations of the standard VD (1), (2) are also mentioned in [4] such as the Laguerre diagram (Power VD), approximate (fuzzy) VD, VD with constraints on the power of generator points, dynamic VD etc. There is also a table of correspondence between a specific VD version with clear parameters and the mathematical model of the OSP problem, the solution of which gives this diagram.

3 Problem Statement

We now turn to the mathematical formulation of the problem of constructing a generalized VD in the presence of fuzzy parameters with the optimal location of a finite number of generator points.

Let Ω be some given bounded set from E_n and let $\tau_1, \tau_2, ..., \tau_N$ be a finite set of generator points in Ω. In cases when the location of points $\tau_1, \tau_2, ..., \tau_N$ in Ω is unknown and they need to be located (selected) in Ω, we enter another VD version on set $\Omega \subset \mathrm{E}_n$, namely, VD of a finite number of points optimally located in a bounded set.

We define [4] ***the Voronoi diagram of a finite number of generator points*** $\tau_1, \tau_2, ..., \tau_N$ ***optimally located in a bounded set*** $\Omega \subset \mathrm{E}_n$ ***with fuzzy parameters*** as a set of Voronoi polytopes

$$Vor(\tau_i) = \left\{ x \in \Omega \subset \mathrm{E}_n : c(x, \tau_i) / w_i + a_i \le c(x, \tau_j) / w_j + a_j,\ i \ne j,\ j = \overline{1, N} \right\}, \quad (3)$$

$$i = \overline{1, N},$$

of points $\tau_1, \tau_2, ..., \tau_N$ where the total weighted distance from points of set Ω to corresponding generator points $\tau_1, ..., \tau_N$ is minimal, so the functional

$$F(\{\tau_1, ..., \tau_N\}) = \sum_{i=1}^{N} \int_{Vor(\tau_i)} (c(x,\ \tau_i) / w_i + a_i) dx \quad (4)$$

gets the minimum value. Here $w_i > 0$ ($i = \overline{1, N}$) are given numbers (weights); $a_i > 0$ ($i = \overline{1, N}$) are given numeric parameters that are fuzzy and will be formalized using the bell-shaped membership function [6, 7]. We will consider parameters a_i, $i = \overline{1, N}$, in (4) as parameters depending on fuzzy factors $\gamma_j, j = \overline{1, q}$ in the form $a_i \equiv a_i(\gamma_1, ..., \gamma_q)$. For example, in terms of the well-known location-allocation problem with fuzzy parameters, parameter a_i ($i = \overline{1, N}$) makes sense of a cost of the i-th enterprise to produce a unit of output and depends on fuzzy factors $\gamma_j, j = \overline{1, 3}$, where γ_1 is a cost of production means (production capacity and means of labor); γ_2 is a cost of labor resources; γ_3 is a cost of natural resources (raw materials and energy).

Note: By specifying values of parameters $w_1,...,w_N$ and a type of function $c(x,\ \tau_i)$ in formula (4), one can obtain various VD versions in the presence of fuzzy parameters with the optimal location of generator points (multiplicatively weighted, additively weighted, etc.).

We present an approach to constructing VD with fuzzy parameters, based on applying the apparatus of the theory of continuous OSP problems [3] and neuro-fuzzy technologies [6-10].

To do this, we first formulate the corresponding problem of optimal partitioning of a set from n-dimensional Euclidean space E_n into subsets with fuzzy parameters in the target functional and with previously unknown coordinates of some specific for each subset points, called "centers" of subsets. Such a problem is a generalization of the problem from [3].

Let Ω be a bounded, Lebesgue measurable set in n-dimensional Euclidean space. We name a set of Lebesgue measurable subsets $\Omega_1,...,\Omega_N$ from $\Omega \subset E_n$ as a possible partitioning of set Ω into its non-intersecting subsets $\Omega_1,...,\Omega_N$ if

$$\bigcup_{i=1}^{N}\Omega_i = \Omega,\ mes\left(\Omega_i \cap \Omega_j\right) = 0,\ i, j = \overline{1,N}\ \ (i \neq j)\ , \qquad (5)$$

where $mes(\cdot)$ is a Lebesgue measure.

We denote [3] by $\sum_{\Omega}^{N}$ the class of all possible partitions of a set Ω into non-intersecting subsets $\Omega_1,...,\Omega_N$ that is

$$\sum_{\Omega}^{N} = \left\{\left(\Omega_1,...,\Omega_N\right):\ \bigcup_{i=1}^{N}\Omega_i = \Omega,\ mes\left(\Omega_i \cap \Omega_j\right) = 0,\ i, j = \overline{1,N}\ \ (i \neq j)\right\}.$$

We introduce the functional

$$F\left(\{\Omega_1,...,\Omega_N\},\{\tau_1,...,\tau_N\}\right) = \sum_{i=1}^{N}\int_{\Omega_i}(c(x,\ \tau_i)/\,w_i + a_i)\ dx \qquad (6)$$

where $c\left(x,\tau_i\right)$ is a given real function bounded on Ω×Ω, measurable by $x = \left(x^{(1)},...,x^{(n)}\right) \in \Omega$ for any fixed $\tau_i = \left(\tau_i^{(1)},...,\tau_i^{(n)}\right) \in \Omega$, for each $i = \overline{1,N}$; $w_i > 0$ ($i = \overline{1,N}$) are given numbers (weights); $a_i > 0$ ($i = \overline{1,N}$) are given fuzzy numeric parameters.

Here and in the following, integrals are understood in the sense of Lebesgue. We assume that the measure of the set of boundary points of subsets Ω_i, $i = \overline{1,N}$, is equal to zero.

We name the following problem as **a continuous problem of optimal partitioning of a set Ω from E_n into its non-intersecting subsets** $\Omega_1,...,\Omega_N$ (some of which may be empty) **with fuzzy parameters in the target functional with finding the coordinates of centers** $\tau_1,\tau_2,...,\tau_N$ **of these subsets**, respectively [3].

Problem A. To find $\min\limits_{\substack{\{\Omega_1,...,\Omega_N\}\in \Sigma_{\Omega}^{N},\\ \{\tau_1,...,\tau_N\}\in \Omega^N}} F\left(\{\Omega_1,...,\Omega_N\},\{\tau_1,...,\tau_N\}\right)$, where the functional $F(\{\Omega_1,...,\Omega_N\},\{\tau_1,...,\tau_N\})$ is defined by (6); the coordinates $\tau_i^{(1)},...,\tau_i^{(n)}$ of centers $\tau_i=(\tau_i^{(1)},...,\tau_i^{(n)})\in\Omega_i$, $i=\overline{1,N}$ are not predefined and should be determined.

We define a pair $(\{\Omega_1^*,..., \Omega_N^*\}, \{\tau_1^*,...,\tau_N^*\})$ that minimizes functional (6) on set $\Sigma_\Omega^N\times\Omega^N$ as an **optimal solution to Problem A**. Wherein partitioning $\{\Omega_1^*,..., \Omega_N^*\}\in\Sigma_\Omega^N$ we define as an **optimal partitioning** of the set $\Omega\subset E_n$ into N subsets, and set $\tau^*=(\tau_1^*,...,\tau_N^*)\in\Omega^N$ of centers $\tau_i^*\in\Omega_i^*$, $i=\overline{1,N}$ as **optimal centers** of the subsets Ω_i^*, $i=\overline{1,N}$ in Problem A.

Let us illustrate a partitioning of $\Omega\subset E_2$ into three subsets Ω_1, Ω_2, Ω_3 with corresponding centers τ_1,τ_2,τ_3 on Fig.1.

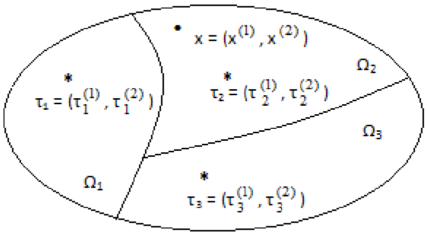

Fig.1. Partitioning of set Ω into three subsets

4 Materials and Methods

To solve Problem A for each fixed a_i ($i=\overline{1,N}$), we introduce the characteristic functions of subsets Ω_i :

$$\lambda_i(x)=\begin{cases}1, & x\in\Omega_i,\\ 0, & x\in\Omega\setminus\Omega_i,\end{cases} \quad i=\overline{1,N},$$

and rewrite Problem A in terms of characteristic functions in the following form.

Problem B. To find $\min\limits_{(\lambda(\cdot),\tau)\in\Gamma\times\Omega^N}\int\limits_\Omega\sum\limits_{i=1}^N(c(x,\tau_i)/w_i+a_i)\,\lambda_i(x)\,dx$,

where $\Gamma=\{\lambda(x)=(\lambda_1(x),...,\lambda_N(x)):\ \sum\limits_{i=1}^N\lambda_{*i}(x)=1$ for a.a. $x\in\Omega$, $\lambda_i(x)=0\vee1$ for a.a. $x\in\Omega$, $i=\overline{1,N}\}$; $\tau=\tau_1,...,\tau_N\in\underbrace{\Omega\times...\times\Omega}_{N}=\Omega^N$.

For Problem B, the theorem was proved in [3], which established the form of an optimal solution $(\lambda_*(\cdot),\tau_*)$.

To remove fuzziness in Problem A, we apply the method of neuro-linguistic identification [6]. Since, as stated above, parameters a_i , $i=\overline{1,N}$, in (6) depend on fuzzy factors γ_j, $j=\overline{1,q}$ in the form $a_i\equiv a_i(\gamma_1,...,\gamma_q)$, we rewrite Problem A with fuzzy parameters as follows:

$$\sum_{i=1}^{N}\int_{\Omega_i}(c(x,\ \tau_i)/w_i+a_i(\gamma_1,...,\gamma_q))dx\rightarrow\min_{\substack{\{\Omega_1,...,\Omega_N\}\in\Sigma_\Omega^N,\\ \{\tau_1,...,\tau_N\}\in\Omega^N}}$$

under condition (5).

To simplify the description of the neuro-linguistic identification method proposed in [6], we denote each parameter a_i, $i=\overline{1,N}$ as a and consider the functional dependence of an output a on the identification object inputs $\gamma_1,...,\gamma_q$ in the form:

$$a=a(\gamma),\quad \gamma=(\gamma_1,...,\gamma_q)\,. \tag{7}$$

For the identification problem, we assume to be known: domains of inputs $\gamma_1,...,\gamma_q$, a range of output a change for (7), and expert-experimental information about the dependency (7) in the form of a sample of data on the object inputs and output. The problem of identifying (recovering) a complex nonlinear dependence of the form (7) is considered as constructing an object model from expert-experimental data on the input-output interrelationships and is usually solved in two stages [7-10]:

– structural identification: the formation of a fuzzy knowledge base about an object and building on its basis a fuzzy object model with several inputs and one output that roughly reproduces the dependence of an output on inputs using the linguistic rules "IF-THEN" generated from experimental data;

– parametric identification (customization): search for such parameters of a fuzzy model that minimize the deviation of model values from experimental ones.

The first stage of the neuro-linguistic identification method (structural identification), which implies constructing a fuzzy model of an object with several inputs and one output, consists of the following blocks: fuzzification, fuzzy inference, defuzzification. At the fuzzification stage, in order to find dependence (7) in an explicit form, we will consider input variables γ_i, $i=\overline{1,q}$ and output variable a as linguistic variables, for the evaluation of which we will use terms from the following term sets [7]:

– $D=\{D_k\}$ is a term-set of variable a, where D_k is the *k*-th linguistic term of variable a, $k=\overline{1,L}$; L is a number of different classes of output a; for each class D_k we define its center $d_k\in D_k$;

– $\Gamma_i=\{\Gamma_{ir}\}$ is a term-set of variable γ_i, where Γ_{ir} is the *r*-th linguistic term of variable γ_i, $i=\overline{1,q}$; $r=\overline{1,t}$; t is a number of terms in term-set Γ_i of variable γ_i.

We obtain values of linguistic terms D_k and Γ_{ir} based on expert-linguistic information about the modeled object. We define each term, like a fuzzy set, by its bell-shaped membership function in the form (10) below. The constructed fuzzy knowledge base based on expert-linguistic information about the modeled object consists of production rules $p_j^k(\gamma_1,\gamma_2,...,\gamma_q)$ and is used when executing a fuzzy inference block (here k ($k=\overline{1,L}$) is an output a class number; j ($j=\overline{1,s_k}$) is a rule number in the k-th class; s_k is a number of rules in the *k*-th class).

As a result of executing the fuzzy inference block, we obtain the membership function $\mu_{D_k}(a)$ of output variable a to class D_k in the form

$$\mu_{D_k}(a) = \begin{cases} \sum_{j=1}^{s_k} p_j^k(\gamma_1, \gamma_2, ..., \gamma_q), & \text{if } \sum_{j=1}^{s_k} p_j^k(\gamma_1, \gamma_2, ..., \gamma_q) \le 1, \\ 1, & \text{otherwise,} \end{cases} \tag{8}$$

where

$$p_j^k(\gamma_1, \gamma_2, ..., \gamma_q) = v_j^k \prod_{i=1}^{q} \mu_{ij}^k(\gamma_i), \quad j = \overline{1, s_k}, \quad k = \overline{1, L}\,, \tag{9}$$

$$\mu_{ij}^k(\gamma_i) = \frac{1}{1 + \left(\frac{\gamma_i - b_{ij}^k}{e_{ij}^k} \right)^2}, \quad i = \overline{1, q}\,, \tag{10}$$

v_j^k is a weight of the j-th rule in the k-th class of output a; $\mu_{ij}^k(\gamma_i)$ is the bell-shaped membership function of variable γ_i to term Γ_{ir} in the k-th class of output a; b_{ij}^k is a maximum coordinate, e_{ij}^k is a concentration coefficient of this membership function ($j = \overline{1, s_k}$; $k = \overline{1, L}$; $i = \overline{1, q}$; $r = \overline{1, t}$).

At the defuzzification stage, in order to obtain an accurate (clear) value of the output variable, we apply the discrete analogue of the center-of-gravity method [7]:

$$a = \frac{\sum_{k=1}^{L} d_k \mu_{D_k}(a)}{\sum_{k=1}^{L} \mu_{D_k}(a)}\,. \tag{11}$$

Thus, we have constructed a fuzzy model of object (7) in the form of relations (8) - (11), which, as noted above, roughly describes the desired relationship (7).

At the second stage of the neuro-linguistic identification method (parametric identification, customization), we will apply the methodology developed in [6] using the N.Z. Shor's r-algorithm [11, 12] to optimize the parameters of model (8) - (11). As a result of solving this optimization problem, we obtain such values v_j^{*k} for the weights of rules (9) and b_{ij}^{*k}, e_{ij}^{*k} for the parameters of membership functions (10), for which the deviation of experimental data from the model ones obtained after customizing the fuzzy object model (7) reaches the minimum value.

In relations (8) - (11), we take for $\mu_{D_k}(a)$, $p_j^k(\gamma_1, \gamma_2, ..., \gamma_q)$, $\mu_{ij}^k(\gamma_i)$ their values, which are calculated at the optimal values v_j^{*k}, b_{ij}^{*k}, e_{ij}^{*k} of the parameters obtained after customization, carried out using the N.Z.Shor's *r*-algorithm [11, 12].

We present below theorem based on the results from [3, 6], which sums up our reasoning and will be used later in the formulation of the algorithm to solve Problem A.

Theorem. *Optimal solution of Problem B has a form* $\lambda_*(x) = (\lambda_{*1}(x), ..., \lambda_{*i}(x), ..., \lambda_{*N}(x))$ *, where for a. a.* $x \in \Omega$

$$\lambda_{*i}(x) = \begin{cases} 1, & c(x,\tau_{*i})/w_i + a_i \le c(x,\tau_{*j})/w_j + a_j, \\ & j = \overline{1,N}\ (j \ne i),\ \text{then } x \in \Omega_{*i}, \qquad\qquad i = \overline{1,N} \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

and $\tau_{*1},\ldots,\ \tau_{*N}$ *are an optimal solution of the problem*

$$G(\tau) = \int\limits_{\Omega} \min_{i=\overline{1,N}} [c(x,\tau_i)/w_i + a_i] dx \to \min,\ \tau \in \Omega^N. \tag{13}$$

Here, each parameter a_i *(* $i = \overline{1,N}$ *) named earlier as output* a *depending on inputs* $\gamma_1,\ldots,\gamma_q$ *in the form* $a = a(\gamma)$, $\gamma = (\gamma_1,\ldots,\gamma_q)$, *is calculated by the following formulas:*

$$a = \frac{\sum_{k=1}^{L} d_k \cdot \mu^*_{D_k}(a)}{\sum_{k=1}^{L} \mu^*_{D_k}(a)}, \tag{14}$$

where

$$\mu^*_{D_k}(a) = \begin{cases} \sum_{j=1}^{s_k} p^{*k}_j(\gamma_1,\gamma_2,\ldots,\gamma_q), & \text{if} \quad \sum_{j=1}^{s_k} p^{*k}_j(\gamma_1,\gamma_2,\ldots,\gamma_q) \le 1, \\ 1, & \text{otherwise,} \end{cases} \tag{15}$$

$$p^{*k}_j(\gamma_1,\gamma_2,\ldots,\gamma_q) = v^{*k}_j \prod_{i=1}^{q} \mu^{*k}_{ij}(\gamma_i), \tag{16}$$

$$\mu^{*k}_{ij}(\gamma_i) = \frac{1}{1+\left(\dfrac{\gamma_i - b^{*k}_{ij}}{e^{*k}_{ij}}\right)^2},\ i = \overline{1,q};\ j = \overline{1,s_k};\ k = \overline{1,L}. \tag{17}$$

Here we present an algorithm to solve Problem A with neuro-linguistic identification of fuzzy parameters a_i, $i = \overline{1,N}$, which is based on the above Theorem and the most efficient (of the known methods of non-smooth optimization) method of generalized gradient descent with space expansion in the direction of the difference of two successive generalized anti-gradients (or the so-called N.Z.Shor's r-algorithm [11, 12]) for solving problem (13), dual to Problem B, with the non-differentiable functional $G(\tau)$.

The idea of generalized gradient descent methods with space expansion is based on successive constructing linear operators that change the metric of space, and on choosing the descent direction corresponding to the anti-gradient in space with a new metric.

In the iterative formula of the r-algorithm [11], which has the form

$$\tau^{k+1} = \tau^k - h_k B^{\tau}_{k+1} \left[B^{\tau}_{k+1}\right]^T g_G\left(\tau^k\right), \quad k = 0,1,2,\ldots, \tag{18}$$

B^{τ}_{k+1} is an operator that maps transformed space into the main space E_N ($B^{\tau}_0 = I$ is identity matrix); h_k is a stepping factor chosen from the condition of function G

minimum towards $B_{k+1}B_{k+1}^{\tau}g_G\left(\tau^k\right)$; $g_G\left(\tau_k\right)$ is a generalized gradient of function $G(\tau)$ at point τ_k.

Here we use the r-algorithm in H-form [11], where H_k is a symmetric matrix such that $H_k = B_k B_k^T$, and the iteration formula (18) is defined as

$$\tau^{k+1} = \tau^k - h_k \frac{H_{k+1}g_G\left(\tau^k\right)}{\sqrt{\left(H_{k+1}g_G\left(\tau^k\right), g_G\left(\tau^k\right)\right)}}, \quad k = 0, 1, 2, ...,$$

where

$$H_{k+1} = H_k + \left(1/\alpha_k^2 - 1\right)\frac{H_k\Delta_k\Delta_k^T H_k}{\left(H_k\Delta_k, \Delta_k\right)}; \quad \Delta_k = g_G\left(\tau^k\right) - g_G\left(\tau^{k-1}\right).$$

Coefficient α_k of space expansion is define as 3. To adjust stepping factor h_k, the adaptive method is used described in [11].

We define the i-th component of generalized gradient vector $g_G^{\tau}\left(\tau\right) = \left(g_G^{\tau_1}\left(\tau\right), ..., g_G^{\tau_i}\left(\tau\right), ..., g_G^{\tau_N}\left(\tau\right)\right)$ of function

$$G(\tau) = \int_{\Omega} \min_{i=1,...,N} [c(x, \tau_i) / w_i + a_i] \, dx,$$

at point τ as

$$g_G^{\tau_i}\left(\tau\right) = \int_{\Omega} g_c^{\tau_i}\left(x; \tau\right) \lambda_i(x) dx, \quad i = \overline{1, N}, \tag{19}$$

where $g_c^{\tau_i}\left(x, \tau\right)$ is the i-th component of vector of generalized gradient $g_C^{\tau}\left(x, \tau\right)$. In formulas (19), $\lambda_i(x)$, $i = \overline{1, N}$, is determined for almost all $x \in \Omega$ as follows:

$$\lambda_i(x) = \begin{cases} 1, & c(x, \tau_i) / w_i + a_i \le c(x, \tau_j) / w_j + a_j, \\ & j = 1, ..., N \ (j \ne i), \text{ then } x \in \Omega_i, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

where a_i, $i = \overline{1, N}$, are calculated by formulas (14) - (17).

4.1 Algorithm

Step 1. We enclose set Ω in n-dimensional parallelepiped Π. Then we cover parallelepiped Π with rectangular grid and define initial approximation $\tau = \tau^{(0)}$. Then we calculate $\lambda^{(0)}\left(x\right)$ in grid nodes by formulas (20) at $\tau = \tau^{(0)}$ taking into account formulas (14)-(17) to find parameter a; we also calculate $g_G\left(\tau\right)$ by formulas (19) at $\lambda\left(x\right) = \lambda^{(0)}\left(x\right)$, $\tau = \tau^{(0)}$, select $h_0 > 0$ and find

$$\tau^1 = \mathrm{P}_{\Pi}\left(\tau^0 - h_0 \frac{H_1 g_G\left(\tau^0\right)}{\sqrt{\left(H_1 g_G\left(\tau^0\right), g_G\left(\tau^0\right)\right)}}\right),$$

where P_{Π} is a projection operator on Π. Then we go to the following step.

Step (*k*+1), k =1, 2,….

1. Calculate $\lambda^{(k)}(x)$ using formula (20), taking into account formulas (14) - (17) for calculating parameter a, with $\tau=\tau^{(k)}$.
2. Find values of $g_G(\tau)$ by formulas (19) with $\lambda(x)=\lambda^{(k)}(x)$, $\tau=\tau^{(k)}$.
3. Carry out the (*k*+1)-th step by the iteration formula [12]

$$\tau^{k+1}=\mathrm{P}_{\Pi}\left(\tau^k-h_k\frac{H_{k+1}g_G\left(\tau^k\right)}{\sqrt{\left(H_{k+1}g_G\left(\tau^k\right),g_G\left(\tau^k\right)\right)}}\right),$$

4. Go to point 5 if

$$\left\|\tau^k-\tau^{k+1}\right\|\le\varepsilon,\quad \varepsilon>0\,, \tag{21}$$

otherwise proceed to the (*k*+2)-th step of algorithm.

5. Assume $\lambda_*(x)=\lambda^{(l)}(x)$, $\tau_*=\tau^{(l)}$, where *l* is an iteration number at which (21) holds true.
6. Calculate $G(\tau)$ from (13) by the formula

$$G(\tau)=\int_{\Omega}\min_{i=1,\dots,N}[c(x,\tau_i)/w_i+a_i]dx\,,$$

with $\tau=\tau_*$ and a_i, $i=\overline{1,N}$, calculated by formulas (14)-(17).

The algorithm is described.

Thus, as a result of solving Problem A by the described algorithm based on the above Theorem, we obtain a set of Voronoi polytopes (3) of generator points τ_i, $i=\overline{1,N}$:

$$\mathrm{Vor}(\tau_i)=\left\{x\in\Omega\subset\mathrm{E}_n:c(x,\tau_i)/w_i+a_i\le c(x,\tau_j)/w_j+a_j,\, i\ne j,\ j=\overline{1,N}\right\},$$

but instead of standard VD (1), in which points $\tau_1,\dots,\tau_N$ are fixed and parameters a_i, $i=\overline{1,N}$, are clear, a solution of the finite-dimensional optimization problem

$$G(\tau)=\int_{\Omega}\min_{i=1,\dots,N}[c(x,\tau_i)/w_i+a_i]dx\to\min,\quad \tau\in\Omega^N=\underbrace{\Omega\times\ldots\times\Omega}_{N},$$

is required to find the coordinates of generator points $\tau_1,\dots,\tau_N$ that are optimally located in $\Omega\subset E_n$. This optimization problem contains a non-differentiable target function $G(\tau)$ and parameters a_i, $i=\overline{1,N}$, reconstructed using the neuro-linguistic identification method [6].

5 Numerical validation

Let us illustrate the algorithm operation to solve a Problem A for Ω from E_2 with fuzzy parameters in the target functional by example of constructing a generalized additive VD with fuzzy parameters and with locating *N*=5 generator points in the set $\Omega=\{(x^{(1)},x^{(2)})\in E_2:0\le x^{(1)}\le 1;0\le x^{(2)}\le 1\}$.

We note first that, as a result of solving Problem A with clear parameters by the described algorithm a generalized additive VD has constructed with located five generator points $\tau_{*1},...,\tau_{*5}$ in Ω and specified clear values of parameters $a_1 = 0,07$; $a_2 = 0,1$; $a_3 = 0,38$; $a_4 = 0,2$; $a_5 = 0$. Function $c(x,\tau_i)$ of distance from a point $x = (x^{(1)}, x^{(2)}) \in \Omega$ to a generator point $\tau_i = (\tau_i^{(1)}, \tau_i^{(2)}) \in \Omega$ was taken as follows: $c(x,\tau_i) = \sqrt{(x^{(1)} - \tau_i^{(1)})^2 + (x^{(2)} - \tau_i^{(2)})^2}$, $i = \overline{1,5}$. As a result of the described algorithm operation, in 46 iterations we also obtained:

– maximum value of functional $G(\tau)$ from (13), equal to 0,26722;

– minimum value of functional F of Problem A, equal to 0,26789.

This VD is presented in Fig. 2, where the solid line indicates boundaries of subsets Ω_i, $i = \overline{1,5}$, and the «•» symbol indicates an optimal coordinates of generator points: τ_{*1} =(0,28439; 0,23794); τ_{*2} =(0,19985; 0,76565); τ_{*4} =(0,81255; 0,14963); τ_{*5} =(0,70583; 0,65526). At that, instead of partitioning into N=5 subsets, the partitioning into four subsets turned out to be optimal: subset Ω_3 turned out to be empty, since the value a_3 is much greater than a_i, $i = \overline{1,5}$. It is easy to see that the boundaries between the different cells of the additive VD presented in Fig. 2 are, as proved in [3], either segments of hyperbola branches, if $a_i \neq a_j$, or straight lines, if $a_i = a_j$, $i, j = \overline{1,5}$. In case $a_i = a_j$, the Voronoi cells are convex polygons.

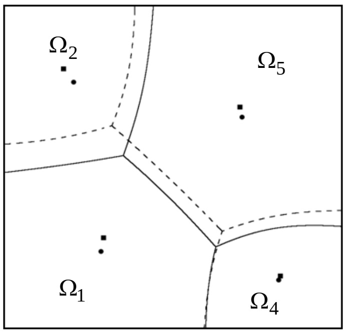

Fig.2. Voronoi diagrams for the model problem

6 Results and Discussion

We now turn to illustrating the approach described in this paper to solve the same problem, but with fuzzy parameters a_i, $i = \overline{1,5}$, in the target functional.

After applying the neuro-linguistic identification method to reconstruct (before customization) parameters a_i, $i = \overline{1,5}$, we got the following their values: $a_1 \approx 0,07493$; $a_2 \approx 0,19432$; $a_3 \approx 0,33743$; $a_4 \approx 0,21138$; $a_5 \approx 0,04007$. Further, as a result of applying the described algorithm to solve Problem A with these reconstructed values of parameters a_i , $i = \overline{1,5}$, in 44 iterations we obtained:

– additive VD with located in Ω generator points $\tau_{*1},...,\tau_{*5}$, that is presented in Fig. 2, where the dashed line indicates boundaries of subsets Ω_i , $i = \overline{1,5}$, and the «■» symbol indicates an optimal coordinates of the corresponding generator points: τ_{*1} =(0,28883; 0,27976); τ_{*2} =(0,17024; 0,80697); τ_{*4} =(0,81631; 0,16100); τ_{*5} =(0,69535; 0,68627);

– maximum value of functional $G(\tau)$ from (13), equal to 0,30075;

– minimum value of functional F of Problem A, equal to 0,30299.

And finally, after reconstructing values of parameters a_i , $i = \overline{1,5}$, using the neuro-linguistic identification method [6] and after the subsequent applying the described above algorithm to solve the considered Problem A with the reconstructed after customizing values of parameters $a_1 = 0,07000$; $a_2 = 0,10000$; $a_3 = 0,38000$; $a_4 = 0,20000$; $a_5 = 0,00000$, in 46 iterations, the same results were obtained (within the specified accuracy $\varepsilon = 0,0001$), as for the case with clear parameters, namely:

– the corresponding VD presented in Fig. 2, where the solid line indicates boundaries of the obtained subsets Ω_i , $i = \overline{1,5}$, and the «●» symbol indicates an optimal coordinates of generator points: τ_{*1} =(0,28439; 0,23794); τ_{*2} =(0,19985; 0,76565); τ_{*4} =(0,81255; 0,14963); τ_{*5} =(0,70583; 0,65526);

– maximum value of functional $G(\tau)$ from (13), equal to 0,26722;

– minimum value of functional F of Problem A, equal to 0,26789.

Comparing the numerical and graphical results of solving this model problem, which are obtained for clear parameters $a_1,...,a_5$ in the target functional and for fuzzy parameters $a_1,...,a_5$ reconstructed using the neuro-linguistic identification method, we see that optimal solutions of these problems coincide with the sufficient degree of accuracy. That is, the proposed approach to construct a generalized VD with fuzzy parameters is reasonable and gives reliable results.

7 Conclusions

A new method and algorithm to construct a generalized VD in the presence of fuzzy parameters with the optimal location of a finite number of generator points in a bounded set of E_n are proposed. The method is based on formulating the corresponding problem of optimal partitioning of a set of E_n into non-intersecting subsets with locating the centers of these subsets with fuzzy parameters in the target functional and with a partitioning quality criterion that provides the corresponding form of VD with fuzzy parameters. The method of solving the above-mentioned OSP problem is based on applying the mathematical apparatus developed in [3] and on using the neuro-linguistic identification method developed in [6] to obviate fuzziness in the OPS problem.

References

1. Voronoi G.F. Collected Works // Izd. Akad. Nauk UkrSSR, Kiev, vol. 2 (1952) [in Russian]
2. Preparata F., Sheimos M. Computational geometry: an introduction. Springer; First Edition edition, 390 p. (1993)
3. Kiseleva E.M., Shor N.Z. Continuous problems of optimal set partitioning: theory, algorithms, applications. Kyiv: Naukova Dumka, 564 p. (2005) [in Russian].
4. Kiseleva E.M., Koriashkina L.S. Theory of continuous optimal set partitioning problems as a universal mathematical formalism for constructing Voronoi diagrams and their generalizations I. Theoretical foundations // Cybernetics and Systems Analysis, vol. 51, № 3, pp. 325-335 (2015). DOI 10.1007/s10559-015-9725-x.
5. Kiseleva E.M., Koriashkina L.S. Theory of continuous optimal set partitioning problems as a universal mathematical formalism for constructing voronoi diagrams and their generalizations. II. Algorithms for constructing Voronoi diagrams based on the theory of optimal set partitioning // Cybernetics and Systems Analysis, vol. 51, № 4, pp. 489-499 (2015). DOI: 10.1007/s10559-015-9740-y.
6. Kiseleva E.M., Pritomanova O.M., Zhuravel S.V. Algorithm for Solving a Continuous Problem of Optimal Partitioning with Neurolinguistic Identification of Functions in Target Functional // Journal of Automation and Information Science, vol. 50, № 3, pp. 1-20 (2018). DOI: 10.1615/JAutomatInfScien.v50.i3.10.
7. Rothstein A. Intellectual technologies of identification: fuzzy sets, genetic algorithms, neural nets. Vinnitsa: UNIVERSUM, 320 p. (1999)
8. Zadeh L.A. The concept of a linguistic variable and its application to approximate reasoning – I // Information Sciences, 8(3), pp.199-249 (1975)
9. Zimmerman H.-J. Fuzzy Sets Theory and Its Applications, 4th ed. Boston: Kluwer Acad. Publ., 514 p. (2001)
10. Borisov V.V., Kruglov V.V., Fedulov A.S. Fuzzy models and networks. Moscow: Hotline-Telecom, 284 p. (2015) [in Russian]
11. Shor, N.Z. Nondifferentiable optimization and polynomial problems. Boston; Dordrecht; London: Kluwer Acad. Publ., 412 p. (1998)
12. Shor N.Z. Minimization methods for non-differentiable functions // Springer series, Computational mathematics. Berlin: Springer-Verlag, vol. 3, 162 p. (1985)
13. Atamtürk A., Nemhauser G.L., Savelsbergh M.W.P. A Combined Lagrangian, Linear Programming and Implication Heuristic for Large-Scale Set Partitioning Problems // Journal of Heuristics, vol. 1, pp. 247-259 (1995)
14. Aurenhammer F., Klein R., Lee D.-T. Voronoi Diagrams and Delaunay Triangulations. World Scientific Pub Co Inc, 337 p. (2013)
15. Bakolas E., Tsiotras P. The Zermelo-Voronoi diagram: A dynamic partition problem // Automatica, 46 (12), pp. 2059-2067 (2010)
16. Balzer M. Capacity-constrained Voronoi diagrams in continuous spaces // The International Symposium on Voronoi Diagrams in Science and Engineering (2009)
17. Jooyandeh Mohammadreza, Mohades Ali, Mirzakhah Maryam. Uncertain Voronoi diagram // Information Processing Letters, 109 (13), 709-712 (2009)

18. Ledoux H., Christopher M. Gold modelling three-dimensional geoscientific fields with the Voronoi diagram and its dual // International Journal of Geographical Information Science, 22 (5), pp. 547-574 (2008)
19. Nishida T., Sugihara K., Kimura M. Stable marker-particle method for the Voronoi diagram in a flow field // Journal of Computational and Applied Mathematics, 202 (2), pp. 377-391 (2007).
20. Okabe A., Boots B, Sugihara K., Chiu S.N. Spatial Tessellations: Concepts and Applications of Voronoi Diagrams // West Sussex, England: John Wiley and Sons Ltd, second ed., 696 p. (2000)