

Improvement of Numerical Solution Smoothness for the Hydrodynamics Problems Modeling on Rectangular Grids

Sofya V. Protsenko,
Don State Technical University,
rab55555@rambler.ru

Alexander E. Chistyakov,
Don State Technical University,
cheese_05@mail.ru

Abstract

The article has been devoted to the problem of improvement real numerical modeling accuracy for the viscous fluid flow between two coaxial half-cylinders on rectangular grids taking into account the filling of cells are used to solve this problem. Approximation of the problem with respect to time is performed on the basis of splitting schemes for physical processes. The simulation was performed on a sequence of condensing computed grids of sizes 11×21 , 21×41 , 41×81 , and 81×161 nodes for the areas of smooth and piecewise rectangular boundaries. The grids taking into account the filling of cells are used to improve the smoothness of the solution. In the case of piecewise rectangular approximation the numerical solution error reaches 70%. The grids taking into account the filling of cells reduce the numerical solution error to 6% for the test problem. The test problem shows that using the grid condensed in each spatial direction by 8 times does not lead to increasing the accuracy solutions whereas the solutions accuracy obtained on the basis proposed approach has significant advantage in accuracy.

1 Introduction

Difference schemes taking into account the degree the filling of cells for solving two-dimensional problems of wave hydrodynamics with dynamically varied geometry of the computational domain were proposed in [Suk12]. The solutions obtained on the basis of these schemes are devoid of defects associated with graded approximation of the boundary. Three-dimensional mathematical model of the movement of the water medium in the Azov Sea was developed on the basis of these schemes [Suk13], the total water depth was 14.2 meters, free surface elevation may be reach 4 meters or more. During calculating the storm surge that occurred in September 2014, in Taganrog bay of Azov sea (wind speed reached 40 m/s), the simulation error was 20 cm with a total over travel of more than 4 meters, the model showed a time lag of about 15 minutes with a total storm interval of about 1000 minutes [Suk18]. The σ -coordinate system is traditionally used in modeling the hydrodynamics of shallow water bodies [Eze00, Mon73, Vas12]. The solutions obtained on these grids have a large error and poorly describe the influence of the bottom relief on nowadays the structure of the currents. The optimal curvilinear

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grids that approximate the boundary ear used as an alternative to rectangular grids, which have low accuracy in the case of direct piecewise rectangular approximation of the boundary [Mur16, Lan86].

When solving the problems of hydrodynamics, inaccuracies arise associated with a stepwise representation of the interface between two media, which can reach 70% of the solution. This error has a high-frequency character. The smoothness of the solution indicates the absence of these assimilations. This scheme has found its application in solving the problems of hydrodynamics of shallow water bodies. The proposed original method is described.

The constructing problem of optimal three-dimensional computational grids remains open nowadays for the 3D regions of common configurations in computational fluid dynamics [Hir81]. Difference schemes accuracy comparison has been discussed in this paper in cases of direct rectangular grids usage and additional involvement of the cell filling function for the Taylor-Couette flow numerical modeling. The proposed method is likely close to Volume Of Fluid (VOF) method [Bun06, Bel75].

2 Statement of the problem

In this work we study the viscous incompressible fluid motion in two-dimensional region between two infinitely long coaxial circular cylinders. We introduce the Cartesian coordinate system xOy perpendicular to the axis of the cylinders. The coordinate system origin coincides with the cylinders' axis. In the section of the cylinder by the plane $x = 0$ defines the field of velocity. It is required to determine the liquid motion. The initial equations for the mathematical description of the fluid dynamics problem are [Bel87, Suk13']:

– Navier-Stokes equation:

$$u'_t + uu'_x + vv'_y = -\frac{P'_x}{\rho} + (\mu v'_x)'_x + (\mu u'_y)'_y, \quad (1)$$

$$v'_t + uv'_x + vv'_y = -\frac{P'_y}{\rho} + (\mu v'_x)'_x + (\mu v'_y)'_y; \quad (2)$$

– the continuity equation for incompressible fluid:

$$u'_x + v'_y = 0. \quad (3)$$

Equations (1)-(3) are considered under the following boundary conditions:

– the flows are defined on the input and output boundaries:

$$u(x, y, t) = U(x, y), v(x, y, t) = V(x, y), P'_n(x, y, t) = 0, \quad (4)$$

– the frictionless and slip conditions are set on the lateral surfaces (in the case $|\tau| = 0$, that is, without friction):

$$P'_n(x, y, t) = 0, u_n(x, y, t) = 0, \rho\mu u'_y(x, y, t) = -\tau_x(t), \rho\mu v'_x(x, y, t) = -\tau_y(t) \quad (5)$$

or sticking condition:

$$P'_n(x, y, t) = 0, u(x, y, t) = 0, v(x, y, t) = 0, \quad (6)$$

where $u = \{u, v\}$ is the water medium velocity vector; (x, y) is Cartesian coordinates, t is time, P is pressure; μ is the turbulent exchange coefficient; ρ is the liquid density; n is the normal vector; τ_x, τ_y are the tangential stress components at the bottom of the liquid.

The wind stress according to the Van Dorn law, is calculated by the formulas [Mon73]:

$$\tau \equiv \{\tau_x, \tau_y\} = \rho C_p (|u|) u |u|. \quad (7)$$

3 Discrete model of hydrodynamics

The computational domain inscribed in a rectangle. For numerical realization of the discrete mathematical model of the formulated wave hydrodynamics problem, uniform grid is introduced:

$$w_h = \{t^n = n\tau, x_i = ih_x, y_j = jh_y; n = \overline{0, \dots, N_t}, i = \overline{0, \dots, N_x}, j = \overline{0, \dots, N_y};$$

$$N_t\tau = T, N_x h_x = l_x, N_y h_y = l_y\},$$

where τ is the time step, h_x, h_y are steps in space, N_t is the step number on the time coordinate, N_x, N_y are spacing steps on the spatial coordinates x and y , respectively.

We use the splitting schemes for physical processes [Sam89, Sam95]. In this case, the solution of the problem (1)-(3) reduces to solving the following system of equations:

$$\frac{u^{n+\sigma} - u^n}{\tau} + uu'_x + vv'_y = (\mu u'_x)'_x + (\mu u'_y)'_y, \quad (8)$$

$$\frac{v^{n+\sigma} - v^n}{\tau} + uv'_x + vv'_y = (\mu v'_x)'_x + (\mu v'_y)'_y, \quad (9)$$

$$P''_{xx} + P''_{yy} = \frac{\rho}{\tau} \left((u^{n+\sigma})'_x + (v^{n+\sigma})'_y \right), \quad (10)$$

$$\frac{u^{n+1} - u^{n+\sigma}}{\tau} = -\frac{P'_x}{\rho}, \quad \frac{v^{n+1} - v^{n+\sigma}}{\tau} = -\frac{P'_y}{\rho}. \quad (11)$$

The calculated cells are rectangles, which may be filled, partially filled, or empty. The cell centers and nodes are separated apart $h_x/2$ and $h_y/2$ on the coordinates x and y , respectively. Fig. 1 (a) shows that the velocity field and pressure are calculated at the tops of the cells. The cells' vertices (i, j) are nodes (i, j) , $(i-1, j)$, $(i, j-1)$, $(i-1, j-1)$.

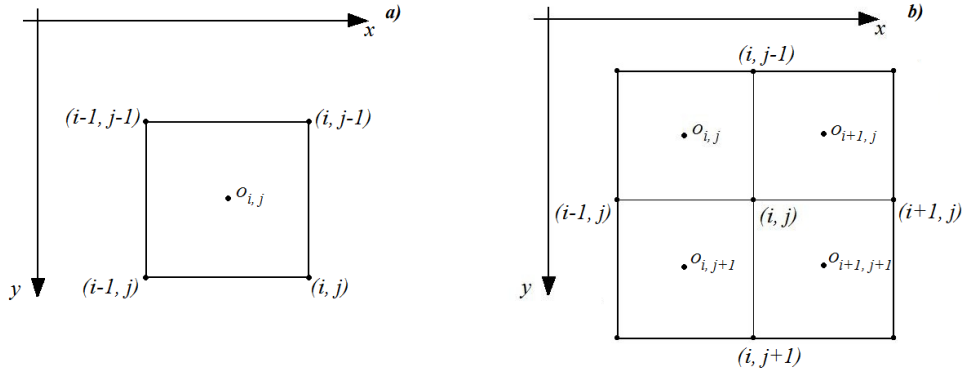


Figure 1: The cell location of the relative to the adjacent nodes and the arrangement of nodes relative to cells

Let us introduce grid value $o_{i,j}$ for the notation of the cell filling. The filling of cells means the value of cell part volume (area) which has been filled with a liquid medium. Fig. 1 (b) shows that in the neighborhood of the node are cells (i, j) , $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$.

We introduce the coefficients k_0, k_1, k_2, k_3, k_4 , describing the filling of regions located in the neighborhood of the cell. The value k_0 characterizes the filling of the region $\Omega_0: x \in (x_{i-1}, x_{i+1}), y \in (y_{j-1}, y_{j+1})$, $k_1 - \Omega_1: x \in (x_i, x_{i+1}), y \in (y_{j-1}, y_{j+1})$, $k_2 - \Omega_2: x \in (x_{i-1}, x_i), y \in (y_{j-1}, y_{j+1})$, $k_3 - \Omega_3: x \in (x_{i-1}, x_{i+1}), y \in (y_j, y_{j+1})$, $k_4 - \Omega_4: x \in (x_{i-1}, x_{i+1}), y \in (y_{j-1}, y_j)$. The filled parts of the regions Ω_m is called D_m , where $m = 0, \dots, 4$. The coefficients k_m can be calculated from the formulas:

$$(k_m)_{i,j} = \frac{S_{D_m}}{S_{\Omega_m}}, (k_0)_{i,j} = \frac{o_{i,j} + o_{i+1,j} + o_{i+1,j+1} + o_{i,j+1}}{4}, (k_1)_{i,j} = \frac{o_{i+1,j} + o_{i+1,j+1}}{2},$$

$$(k_2)_{i,j} = \frac{o_{i,j} + o_{i,j+1}}{2}, (k_3)_{i,j} = \frac{o_{i+1,j+1} + o_{i,j+1}}{2}, (k_4)_{i,j} = \frac{o_{i,j} + o_{i+1,j}}{2}.$$

The boundary conditions for the first subproblem of wave hydrodynamics (8), (9) take form:

$$\begin{aligned} u'_x(x, y, t) &= \alpha_{u,x}u + \beta_{u,x}, v'_x(x, y, t) = \alpha_{v,x}v + \beta_{v,x}, \\ u'_y(x, y, t) &= \alpha_{u,y}u + \beta_{u,y}, v'_y(x, y, t) = \alpha_{v,y}v + \beta_{v,y}. \end{aligned} \quad (12)$$

We integrate equation (8) over the region D_0 and use the property of linearity of the integral, as a result of which we obtain:

$$\iint_{D_0} \frac{u^{n+\sigma} - u^n}{\tau} dx dy + \iint_{D_0} uu'_x dx dy + \iint_{D_0} vv'_y dx dy = \iint_{D_0} (\mu u'_x)'_x dx dy + \iint_{D_0} (\eta u'_y)'_y dx dy. \quad (13)$$

After calculating separately each of the integrals we obtain:

$$\iint_{D_0} \frac{u^{n+\sigma} - u^n}{\tau} dxdy \simeq (k_0)_{i,j} \iint_{\Omega_0} \frac{u^{n+\sigma} - u^n}{\tau} dxdy = (k_0)_{i,j} \frac{u_{i,j}^{n+\sigma} - u_{i,j}^n}{\tau} h_x h_y. \quad (14)$$

The second integral in expression (13) may be written in the form:

$$\iint_{D_0} uu'_x dxdy = \iint_{D_1} uu'_x dxdy + \iint_{D_2} uu'_x dxdy \simeq (k_1)_{i,j} \iint_{\Omega_1} uu'_x dxdy + (k_2)_{i,j} \iint_{\Omega_2} uu'_x dxdy.$$

Calculating the integrals over the regions Ω_1 and Ω_2 , we obtain

$$\iint_{D_0} uu'_x dxdy = \frac{(k_1)_{i,j} u_{i+1/2,j} h_y (u_{i+1,j} - u_{i,j}) + (k_2)_{i,j} u_{i-1/2,j} h_y (u_{i,j} - u_{i-1,j})}{2}. \quad (15)$$

We calculate the integral on the right-hand side of expression (13):

$$\iint_{D_0} (\mu u'_x)'_x dxdy = \iint_{D_1} (\mu u'_x)'_x dxdy + \iint_{D_2} (\mu u'_x)'_x dxdy.$$

In the last equality, let us assume that $S_{D_1} > S_{D_2}$, where we select from the region D_1 fragment $D_{1,2}$, adjacent to the region D_2 , and $S_{D_2} = S_{D_{1,2}}$ (Fig. 3).

$$\begin{aligned} \iint_{D_0} (\mu u'_x)'_x dxdy &= \iint_{D_1/D_{1,2}} (\mu u'_x)'_x dxdy + \iint_{D_{1,2} \cup D_2} (\mu u'_x)'_x dxdy \simeq \\ &\simeq \left((k_1)_{i,j} - (k_2)_{i,j} \right) \iint_{\Omega_1} (\mu u'_x)'_x dxdy + (k_2)_{i,j} \iint_{\Omega_0} (\mu u'_x)'_x dxdy. \end{aligned}$$

As a result, we get:

$$\begin{aligned} \iint_{D_0} (\mu u'_x)'_x dxdy &\simeq \left((k_1)_{i,j} \mu_{i+1/2,j} \frac{u_{i+1,j} - u_{i,j}}{h_x} - (k_2)_{i,j} \mu_{i-1/2,j} \frac{u_{i,j} - u_{i-1,j}}{h_x} - \right. \\ &\left. \left((k_1)_{i,j} - (k_2)_{i,j} \right) \mu_{i,j} (\alpha_{u,x} u_{i,j} + \beta_{u,x}) \right) h_y. \end{aligned} \quad (16)$$

In case, if $S_{D_2} > S_{D_1}$, the result will be similar. Substituting into equation (13), the expressions (14) - (16), we readily obtain:

$$\begin{aligned} &(k_0)_{i,j} \frac{u_{i,j}^{n+\sigma} - u_{i,j}^n}{\tau} h_x h_y + \left((k_1)_{i,j} u_{i+1/2,j} h_y (u_{i+1,j} - u_{i,j}) + (k_2)_{i,j} u_{i-1/2,j} h_y (u_{i,j} - u_{i-1,j}) \right) / 2 + \\ &+ \left((k_3)_{i,j} v_{i,j+1/2} h_x (u_{i,j+1} - u_{i,j}) + (k_4)_{i,j} v_{i,j-1/2} h_x (u_{i,j} - u_{i,j-1}) \right) / 2 = \\ &= \left((k_1)_{i,j} \mu_{i+1/2,j} \frac{u_{i+1,j} - u_{i,j}}{h_x} - (k_2)_{i,j} \mu_{i-1/2,j} \frac{u_{i,j} - u_{i-1,j}}{h_x} - \right. \\ &\left((k_1)_{i,j} - (k_2)_{i,j} \right) \mu_{i,j} (\alpha_{u,x} u_{i,j} + \beta_{u,x}) \right) h_y + \left((k_3)_{i,j} \eta_{i,j+1/2} \frac{u_{i,j+1} - u_{i,j}}{h_y} - \right. \\ &\left. - (k_4)_{i,j} \eta_{i,j-1/2} \frac{u_{i,j} - u_{i,j-1}}{h_y} - \left((k_3)_{i,j} - (k_4)_{i,j} \right) \eta_{i,j} (\alpha_{u,y} u_{i,j} + \beta_{u,y}) \right) h_x. \end{aligned} \quad (17)$$

If we divide the obtained expression by the area of the cell $h_x h_y$ we are coming to:

$$\begin{aligned} &(k_0)_{i,j} \frac{u_{i,j}^{n+\sigma} - u_{i,j}^n}{\tau} + (k_1)_{i,j} u_{i+1/2,j} \frac{u_{i+1,j} - u_{i,j}}{2h_x} + (k_2)_{i,j} u_{i-1/2,j} \frac{u_{i,j} - u_{i-1,j}}{2h_x} + \\ &+ (k_3)_{i,j} v_{i,j+1/2} \frac{u_{i,j+1} - u_{i,j}}{2h_y} + (k_4)_{i,j} v_{i,j-1/2} \frac{u_{i,j} - u_{i,j-1}}{2h_y} = (k_1)_{i,j} \mu_{i+1/2,j} \frac{u_{i+1,j} - u_{i,j}}{h_x^2} - \end{aligned}$$

$$\begin{aligned}
& - (k_2)_{i,j} \mu_{i-1/2,j} \frac{u_{i,j} - u_{i-1,j}}{h_x^2} - \left((k_1)_{i,j} - (k_2)_{i,j} \right) \mu_{i,j} \frac{\alpha_{u,x} u_{i,j} + \beta_{u,x}}{h_x} + \\
& + (k_3)_{i,j} \eta_{i,j+1/2} \frac{u_{i,j+1} - u_{i,j}}{h_y^2} - (k_4)_{i,j} \eta_{i,j-1/2} \frac{u_{i,j} - u_{i,j-1}}{h_y^2} - \left((k_3)_{i,j} - (k_4)_{i,j} \right) \eta_{i,j} \frac{\alpha_{u,y} u_{i,j} + \beta_{u,y}}{h_y}.
\end{aligned} \tag{18}$$

In a similar way, one can obtain discrete analogs for equations (9)-(12). In order to simplify the recording of equations, a mask of the boundary condition $m_{i,j}$ is introduced. The parameter $m_{i,j}$ takes the value 1 if the node (i, j) belongs to the boundary nodes set located in the border region where slip occurs with friction, otherwise $m_{i,j} = 0$. The discrete model of the hydrodynamic problem may be represented by the following grid equations [Kon02]:

– for the component of the velocity vector $u_{i,j}$ under slip condition:

$$\begin{aligned}
& (k_0)_{i,j} \frac{u_{i,j}^{n+\sigma} - u_{i,j}^n}{\tau} + (k_1)_{i,j} u_{i+1/2,j}^n \frac{u_{i+1,j}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{2h_x} + (k_2)_{i,j} u_{i-1/2,j}^n \frac{u_{i,j}^{n+\sigma/2} - u_{i-1,j}^{n+\sigma/2}}{2h_x} + \\
& + (k_3)_{i,j} v_{i,j+1/2}^n \frac{u_{i,j+1}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{2h_y} + (k_4)_{i,j} v_{i,j-1/2}^n \frac{u_{i,j}^{n+\sigma/2} - u_{i,j-1}^{n+\sigma/2}}{2h_y} = \\
& = (k_1)_{i,j} \mu_{i+1/2,j} \frac{u_{i+1,j}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{h_x^2} - (k_2)_{i,j} \mu_{i-1/2,j} \frac{u_{i,j}^{n+\sigma/2} - u_{i-1,j}^{n+\sigma/2}}{h_x^2} + \\
& + (k_3)_{i,j} \mu_{i,j+1/2} \frac{u_{i,j+1}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{h_y^2} - (k_4)_{i,j} \mu_{i,j-1/2} \frac{u_{i,j}^{n+\sigma/2} - u_{i,j-1}^{n+\sigma/2}}{h_y^2} - \left| (k_3)_{i,j} - (k_4)_{i,j} \right| \frac{\tau_x}{\rho h_y} m_{i,j};
\end{aligned}$$

under sticking condition:

$$\begin{aligned}
& \frac{u_{i,j}^{n+\sigma} - (k_0)_{i,j} u_{i,j}^n}{\tau} + (k_{1,2})_{i,j} u_{i+1/2,j}^n \frac{u_{i+1,j}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{2h_x} + (k_{1,2})_{i,j} u_{i-1/2,j}^n \frac{u_{i,j}^{n+\sigma/2} - u_{i-1,j}^{n+\sigma/2}}{2h_x} + \\
& + (k_{3,4})_{i,j} v_{i,j+1/2}^n \frac{u_{i,j+1}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{2h_y} + (k_{3,4})_{i,j} v_{i,j-1/2}^n \frac{u_{i,j}^{n+\sigma/2} - u_{i,j-1}^{n+\sigma/2}}{2h_y} = \\
& = (k_{1,2})_{i,j} \mu_{i+1/2,j} \frac{u_{i+1,j}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{h_x^2} - (k_{1,2})_{i,j} \mu_{i-1/2,j} \frac{u_{i,j}^{n+\sigma/2} - u_{i-1,j}^{n+\sigma/2}}{h_x^2} + \\
& + (k_{3,4})_{i,j} \mu_{i,j+1/2} \frac{u_{i,j+1}^{n+\sigma/2} - u_{i,j}^{n+\sigma/2}}{h_y^2} - (k_{3,4})_{i,j} \mu_{i,j-1/2} \frac{u_{i,j}^{n+\sigma/2} - u_{i,j-1}^{n+\sigma/2}}{h_y^2}, \\
& (k_{1,2})_{i,j} = \min \left\{ (k_1)_{i,j}, (k_2)_{i,j} \right\}, \quad (k_{3,4})_{i,j} = \min \left\{ (k_3)_{i,j}, (k_4)_{i,j} \right\};
\end{aligned}$$

for the velocity vector $v_{i,j}$ component under slip conditions:

$$\begin{aligned}
& (k_0)_{i,j} \frac{v_{i,j}^{n+\sigma} - v_{i,j}^n}{\tau} + (k_1)_{i,j} u_{i+1/2,j}^n \frac{v_{i+1,j}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{2h_x} + (k_2)_{i,j} u_{i-1/2,j}^n \frac{v_{i,j}^{n+\sigma/2} - v_{i-1,j}^{n+\sigma/2}}{2h_x} + \\
& + (k_3)_{i,j} v_{i,j+1/2}^n \frac{v_{i,j+1}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{2h_y} + (k_4)_{i,j} v_{i,j-1/2}^n \frac{v_{i,j}^{n+\sigma/2} - v_{i,j-1}^{n+\sigma/2}}{2h_y} = \\
& = (k_1)_{i,j} \mu_{i+1/2,j} \frac{v_{i+1,j}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{h_x^2} - (k_2)_{i,j} \mu_{i-1/2,j} \frac{v_{i,j}^{n+\sigma/2} - v_{i-1,j}^{n+\sigma/2}}{h_x^2} + \\
& + (k_3)_{i,j} \mu_{i,j+1/2} \frac{v_{i,j+1}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{h_y^2} - (k_4)_{i,j} \mu_{i,j-1/2} \frac{v_{i,j}^{n+\sigma/2} - v_{i,j-1}^{n+\sigma/2}}{h_y^2} - \left| (k_1)_{i,j} - (k_2)_{i,j} \right| \frac{\tau_y}{\rho h_x} m_{i,j};
\end{aligned}$$

under sticking condition:

$$\frac{v_{i,j}^{n+\sigma} - (k_0)_{i,j} v_{i,j}^n}{\tau} + (k_{1,2})_{i,j} u_{i+1/2,j}^n \frac{v_{i+1,j}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{2h_x} + (k_{1,2})_{i,j} u_{i-1/2,j}^n \frac{v_{i,j}^{n+\sigma/2} - v_{i-1,j}^{n+\sigma/2}}{2h_x} +$$

$$\begin{aligned}
& + (k_{3,4})_{i,j} v_{i,j+1/2}^n \frac{v_{i,j+1}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{2h_y} + (k_{3,4})_{i,j} v_{i,j-1/2}^n \frac{v_{i,j}^{n+\sigma/2} - v_{i,j-1}^{n+\sigma/2}}{2h_y} = \\
& = (k_{1,2})_{i,j} \mu_{i+1/2,j} \frac{v_{i+1,j}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{h_x^2} - (k_{1,2})_{i,j} \mu_{i-1/2,j} \frac{v_{i,j}^{n+\sigma/2} - v_{i-1,j}^{n+\sigma/2}}{h_x^2} + \\
& + (k_{3,4})_{i,j} \mu_{i,j+1/2} \frac{v_{i,j+1}^{n+\sigma/2} - v_{i,j}^{n+\sigma/2}}{h_y^2} - (k_{3,4})_{i,j} \mu_{i,j-1/2} \frac{v_{i,j}^{n+\sigma/2} - v_{i,j-1}^{n+\sigma/2}}{h_y^2};
\end{aligned}$$

– for calculation the pressure field:

$$\begin{aligned}
& (k_1)_{i,j} \frac{P_{i+1,j} - P_{i,j}}{h_x^2} - (k_2)_{i,j} \frac{P_{i,j} - P_{i-1,j}}{h_x^2} + (k_3)_{i,j} \frac{P_{i,j+1} - P_{i,j}}{h_y^2} - (k_4)_{i,j} \frac{P_{i,j} - P_{i,j-1}}{h_y^2} = \\
& = \frac{\rho}{\tau} \left(\frac{(k_1)_{i,j} u_{i+1/2,j}^{n+\sigma} - (k_2)_{i,j} u_{i-1/2,j}^{n+\sigma}}{h_x} + \frac{(k_2)_{i,j} - (k_1)_{i,j}}{h_x} U_{i,j} + \right. \\
& \left. + \frac{(k_3)_{i,j} v_{i,j+1/2}^{n+\sigma} - (k_4)_{i,j} v_{i,j-1/2}^{n+\sigma}}{h_y} + \frac{(k_4)_{i,j} - (k_3)_{i,j}}{h_y} V_{i,j} \right);
\end{aligned}$$

– equations to refine the velocity field by pressure:

$$\begin{aligned}
(k_0)_{i,j} \frac{u_{i,j}^{n+1} - u_{i,j}^{n+\sigma}}{\tau} & = - \left((k_1)_{i,j} \frac{P_{i+1,j}^{n+1} - P_{i,j}^{n+1}}{2h_x \rho} + (k_2)_{i,j} \frac{P_{i,j}^{n+1} - P_{i-1,j}^{n+1}}{2h_x \rho} \right), \\
(k_0)_{i,j} \frac{v_{i,j}^{n+1} - v_{i,j}^{n+\sigma}}{\tau} & = - \left((k_3)_{i,j} \frac{P_{i,j+1}^{n+1} - P_{i,j}^{n+1}}{2h_y \rho} + (k_4)_{i,j} \frac{P_{i,j}^{n+1} - P_{i,j-1}^{n+1}}{2h_y \rho} \right).
\end{aligned}$$

It is shown that the order of approximation of the system of equations is $O(\tau + h_x^2 + h_y^2)$. The sufficient condition for the stability of the scheme for the method of pressure correction is determined on the basis of the grid maximum principle [Suk12'] with spacing values restrictions: $h_x < |\frac{2\mu}{u}|$, $h_y < |\frac{2\mu}{v}|$ or $Re \leq 2N$, where $Re = u \cdot l/\mu$ is the Reynolds number, u is the velocity of the aquatic environment, l is the characteristic size of the region, μ is the turbulent exchange coefficient.

To solve the grid equations obtained, an adaptive modified alternating-triangular method of variational type was applied, which is advanced variant of SSOR method.

4 Taylor-Couette flow

Let us consider the steady flow of fluid between two infinitely long coaxial circular cylinders

$$uu'_x + vv'_y = -\rho^{-1}P'_x + \mu\Delta u, uv'_x + vv'_y = -\rho^{-1}P'_y + \mu\Delta v, r_1 \leq r \leq r_2, r = \sqrt{x^2 + y^2}.$$

Suppose, on the internal side, the rotation speed is $|u|_{r=r_1} = u_1$, on the external side, the rotation speed is $|u|_{r=r_2} = u_2$. The polar coordinate system was introduced to solve the problem ($x = r \cos \theta$, $y = r \sin \theta$)

$$\begin{aligned}
u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} & = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (ru_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \\
u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta u_r}{r} & = -\frac{1}{r\rho} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (ru_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right).
\end{aligned}$$

Taking into account that $v_r = 0$, $v_\theta = v_\theta(r)$ and $P = P(r)$, we obtain:

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{u_\theta^2}{r}, \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (ru_\theta)}{\partial r} \right) = 0.$$

The analytical solution of this system of equations is:

$$u_\theta(r) = A_1 r + A_2/r, P(r) = P(r_1) + \rho \int_{r_1}^r (u_\theta^2/r) dr.$$

To compare the results of numerical calculations with the analytical solution, we take $r_1 = 5 \text{ m}$, $r_2 = 10 \text{ m}$, $u_1 = 1 \text{ m/s}$, $u_2 = 0.5 \text{ m/s}$. In this case, the analytical solution takes the form

$$u_\theta(r) = 5/r, P(r) = P(r_1) - 12.5\rho/r^2 + \rho/2.$$

The analytical solution in the Cartesian coordinate system takes the form

$$u(x, y) = -\frac{5y}{x^2 + y^2}, v(x, y) = \frac{5x}{x^2 + y^2}, P(x, y) = P(r_1) - \frac{12.5\rho}{x^2 + y^2} + \rho/2.$$

5 Results of numerical experiments

The problem of finding the numerical flow of a viscous fluid between two coaxial cylinders ($x \geq 0$) is considered. The inside cylinder radius is $r_1 = 5 \text{ m}$. The outside cylinder radius is $r_2 = 10$. The calculated domain is inscribed in a rectangle with dimensions $10 \times 20 \text{ m}$ ($0 \leq x \leq 10$, $-10 \leq y \leq 10$). In the section of the cylinder by the plane $x = 0$ sets the velocity field $u(0, y) = -5/y \text{ m/s}$, $v(0, y) = 0 \text{ m/s}$. In all other grid nodes, the velocity field is calculated. On the inside and outside walls of the cylinder, the conditions for slip and non-flow are specified.

Defects of numerical solutions are most clearly visible on coarse grids. We describe the parameters of a coarse grid. The steps in the spatial directions are 1 m , the time step is 0.1 s , the mesh size is 21×11 knots, the length of the counting interval is 10 s , the density is $\rho = 1000 \text{ kg/m}^3$, the turbulent exchange coefficient is $\mu = 1 \text{ m}^2/\text{s}$. Fig. 2 shows the contents of an array describing the degree of filling of cells in the case of using the grid of 21×11 nodes.

	0	1	2	3	4	5	6	7	8	9
0	0.983	0.883	0.678	0.362	0.03	0	0	0	0	0
1	1	1	1	1	0.894	0.344	0	0	0	0
2	1	1	1	1	1	1	0.59	0.01	0	0
3	1	1	1	1	1	1	1	0.59	0	0
4	1	1	1	1	1	1	1	1	0.344	0
5	0.034	0.24	0.683	1	1	1	1	1	0.894	0.03
6	0	0	0	0.453	1	1	1	1	1	0.362
7	0	0	0	0	0.683	1	1	1	1	0.678
8	0	0	0	0	0.24	1	1	1	1	0.883
9	0	0	0	0	0.034	1	1	1	1	0.983
10	0	0	0	0	0.034	1	1	1	1	0.983
11	0	0	0	0	0.24	1	1	1	1	0.883
12	0	0	0	0	0.683	1	1	1	1	0.678
13	0	0	0	0.453	1	1	1	1	1	0.362
14	0.034	0.24	0.683	1	1	1	1	1	0.894	0.03
15	1	1	1	1	1	1	1	1	0.344	0
16	1	1	1	1	1	1	1	0.59	0	0
17	1	1	1	1	1	1	0.59	0.01	0	0
18	1	1	1	1	0.894	0.344	0	0	0	0
19	0.983	0.883	0.678	0.362	0.03	0	0	0	0	0

Figure 2: The values of the filling of cells for the grid of 21×11 nodes

Fig. 3 (a), (b) shows the numerical solution of the problem of fluid flow between two coaxial cylinders. The color shows the flow of fluid $|k_0 u|$.

Fig. 3 (c) illustrates that the solution of the problem of fluid flow between two coaxial cylinders, obtained on grids that take into account the filling of the cells, is sufficiently smooth. Fig. 3 (d) shows the solution with defects associated with piecewise rectangular approximation of the interface between two media.

Fig. 4 and 5 show the errors in the numerical solution of the problem of fluid flow between two coaxial cylinders on grids taking into account the filling of the cells (in case of a smooth boundary) and on grids with piecewise rectangular approximation of the boundary. For numerical investigation of the accuracy of the proposed schemes, a solution is found on a sequence of condensing grids. Fig. 8 presents the numerical solution of the initial problem of fluid flow between two coaxial cylinders on more detailed grids of sizes 21×41 and 41×81 knots.

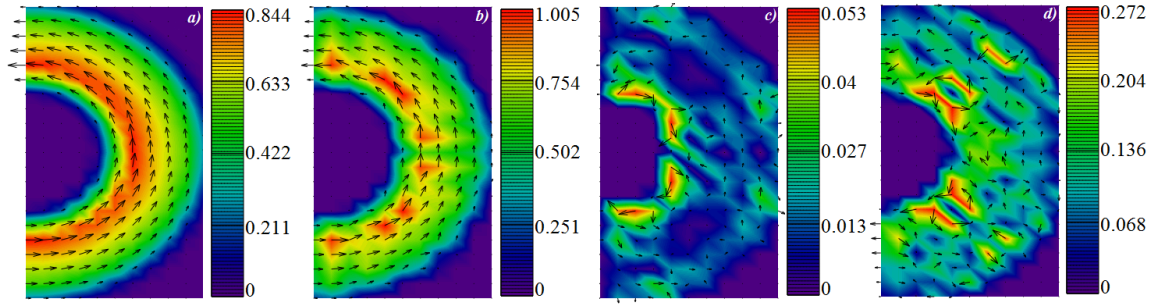


Figure 3: Numerical solution of the problem: a) in case of partial filling of cells, b) in case of piecewise rectangular interface between two media

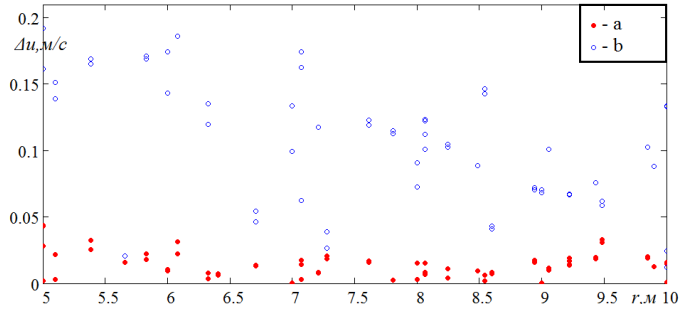


Figure 4: The dependence of the error from the radius: a) in case of smooth boundary, b) in case of step boundary

Fig. 6 shows the error values of the numerical solution of the fluid flow problem, depending on the radius (circles indicate the error in case of a smooth boundary, circles indicate the error in the case of a step boundary).

Fig. 4, 6 show that the increase in the size of the calculated grids for the problem of flow of the aqueous medium does not lead to an increase in the accuracy in case of piecewise rectangular approximation of the boundary, but to a decrease in the linear dimensions of the border region where the solutions of the solution associated with rough approximation of the boundary are manifested. It should also be noted that when using grids taking into account the filling of cells, the error in the numerical solution of model hydrodynamic problems caused by the approximation of the boundary does not exceed 6% of the solution of the problem.

Table 1 presents the error values of the numerical solution of the fluid flow problem between two coaxial cylinders obtained from a sequence of condensing computed grids 11×21 , 21×41 , 41×81 , and 81×161 nodes in case of a smooth and stepped boundary.

Table 1: The error in the solution of the problem of fluid flow between two cylinders

Grid dimensions	11×21	21×41	41×81	81×161
The maximum error value in the case of a smooth boundary, m / s	<i>0.053</i>	<i>0.052</i>	<i>0.058</i>	<i>0.056</i>
The average error value in the case of a smooth boundary, m / s	<i>0.023</i>	<i>0.012</i>	<i>0.006</i>	<i>0.003</i>
The maximum error value in the case of a stepped boundary, m / s	<i>0.272</i>	<i>0.731</i>	<i>0.717</i>	<i>0.75</i>
The average error value in the case of a stepped boundary, m / s	<i>0.165</i>	<i>0.132</i>	<i>0.069</i>	<i>0.056</i>

The analysis of the error calculating results of the numerical solution of the problem of fluid flow between two cylinders on the sequence of condensing grids presented in Table 1 allows us to conclude that the use of difference schemes taking into account the filling of cells is effective. The grid splitting by 8 times in each of the spatial directions does not lead to an increase in the accuracy that solutions obtained on grids taking into account the filling of the cells possess.

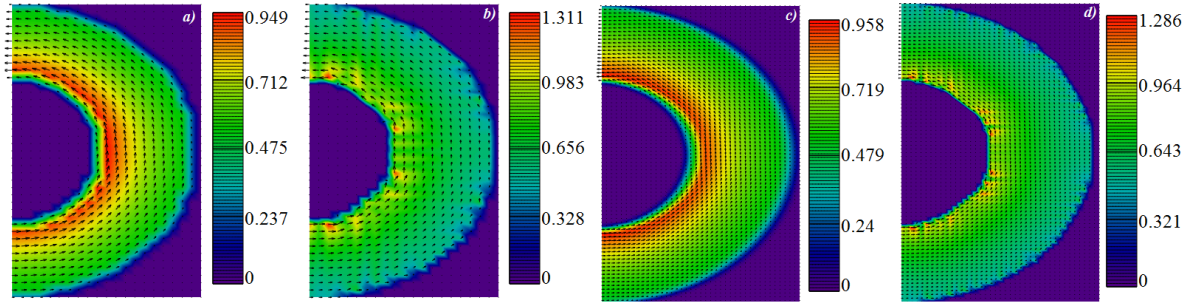


Figure 5: The numerical solution of the problem: a), c) in case of using partial filling of cells; b), d) in case of stepped interface between two media; a), b) dimensions of the computational grid 21×41 knots; c), d) dimensions of the computational grid 41×81 knots

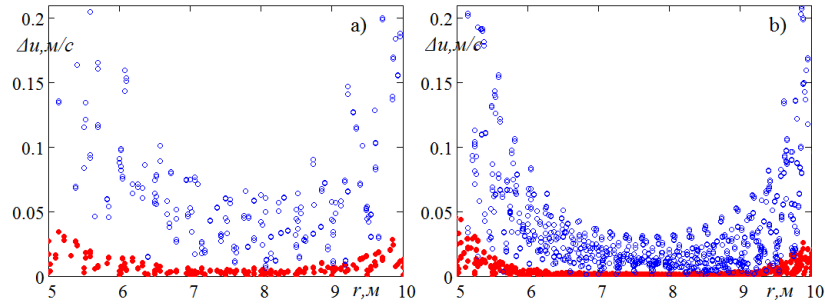


Figure 6: The dependence of the error in the solution of the problem of fluid flow between two cylinders from the radius on a grid of dimensions: a) 21×41 knots, b) 41×81 knots

6 Conclusion

The paper considers the problem of searching the numerical flow of a viscous fluid between two coaxial half-cylinders. Analytic solution describing the Taylor-Couette flow is used as a standard to evaluate the accuracy of the numerical solution of hydrodynamic problems. The simulation was performed on a sequence of condensing computed grids of sizes 11×21 , 21×41 , 41×81 , and 81×161 nodes in cases of smooth and piecewise rectangular boundaries. To improve the solution smoothness the we used grids taking into account the filling of the cells. When solving the hydrodynamics of shallow reservoirs, rectangular meshes are mainly used. This is due to the large difference in step lengths in horizontal and vertical directions.

In the case of piecewise rectangular approximation the error of numerical solution reaches 70%. The grids taking into account the filling of cells reduce the numerical solution error to 6%. It is shown that crushing the grid by 8 times in each spatial direction does not lead to increasing the accuracy solutions whereas the solutions accuracy obtained on grids taking into account the filling of cells significantly increases.

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