Simulation a Modified Erlang Loss System with Multi-type Servers and Multi-type Customers*

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Abstract. We consider a modified multiserver Erlang loss system with two-priority classes of customers: the first priority customers (class-1) are lost if find all servers busy, while the second priority customers (class-2) join an infinite capacity queue if find all servers busy. A new feature of this system is that a *multi-type* servers assignment for the first priority customers is allowed. We assume Poisson inputs and general service times for both classes. We show how the product form of class-1 stationary probabilities can be used to obtain the stability condition of the whole system. Also we perform simulation to confirm theoretical results.

Keywords: Modified Erlang System · Two-Priority Customers · Multi-type Customers · Multi-type Servers · Simulation

1 Introduction

Internet traffic has been explosively increased in recent years and the most likely it will grow in a future. Increased using of the smartphones, tablet and laptop computers, smart watches etc. causes a spectrum shortage problems in wireless networks. One of the way to solve this problem is to use cognitive wireless networks [3]. There can be two classes of users in these networks: primary (licensed) users and secondary (un-licensed) users. Secondary users may use the bandwidths only if they do not interfere primary users and, in particular, if primary users are not present. Primary users can interrupt transmissions of secondary users, if there are no free channels. In this case secondary user evacuates to the head of the buffer. When one of the channels becomes available, secondary users can resume transmissions. We also assume that different channels in general have different transmission rates and can accept a limited set of classes of the primary users. This assumption is related to the notion of flexible servers [9,10]. This notion means that some service capacity can be transferred from one pool of servers to another to satisfy different requirements. We also mention the concept of cross-trained servers [11,12,13], where one pool of servers can serve a limited set of customer types, while a second pool has been trained to serve

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all types of customers. Models with flexible servers and multiclass multiserver setting often use for the purpose of finding an optimal allocation by minimizing a cost function [14,9,15].

Based on this motivation we will study the following queueing system. First, following work [1], we consider a modified Erlang loss system with c identical servers and two classes of customers: the first priority customers (class-1) are lost if find all servers busy, while the second priority customers (class-2) join an infinite capacity queue if find all servers busy. Class-1 customers have absolute priority over class-2 customers meaning that the transmission of a class-2 customer might be interrupted by a class-1 customer. Interrupted class-1 customers evacuate to the head of the buffer and resume their transmission as soon as a server is available.

In the present research we study an extension of this system which in fact is a combination of described modified Erlang system and another multi-class multi-server Erlang-type system introduced and studied in the paper [2]. In this system each server is served by only a *limited set* of customer classes and different classes of customers in general have different arrival rates. Also each server in general has different service rates. The customer assignment probabilities appear in this system: for each customer class and for each set of idle servers the assignment probability of this class customers to each available server must be given. The analysis performed in the work [2] shows that one can choose the assignment probabilities in such a way that the system becomes *reversible* (that is described by a *reversible Markov process*) and in this case the stationary probabilities $\{\mathsf{P}_i\}$ can be found in an explicit form. These stationary probabilities will appear in stability condition of our system (1) and mean the probabilities that the system has i busy servers by class-1 customers.

In this work we combine these systems in such a way that the first priority customers (class-1) considered in the work [1] are divided into I *subclasses* which are the classes of customers in the paper [2]. We note that the stationary probabilities in this system are the same as in [2], since the second priority customers do not affect the assignment and the service of the first priority customers. We show that stability condition of the system described in [1] is also the stability condition of our extended system provided the service rates are the same for all servers, as in classical multi-server systems (identical servers). In this case one can use found stationary probabilities $\{\mathsf{P}_i\}$ to obtain stability condition (1) (see below) of our extended system. We note that this condition is correct if the assignment probabilities of class-1 customers satisfy system equations (12) (see below). In turn system (12) is adapted from [2].

A similar model was studied by Akutsu and Phung-Duc [3] where primary customer first sense the channels before occupying them. In this work Poisson arrivals and exponential service times of all classes of users are assumed. The stability condition is suggested and verified by simulation in [3]. Other queueing models of cognitive radio networks could be found in the papers [4,6,7,8].

The rest of paper is the following. In Section 2 we describe an extended system and show the stability condition of this system. In Section 3 we present some preliminary results from the work [2]. In Section 4 we construct an example of the extended system to show in detail how to find the assignment probabilities (which must satisfy some balance equations) and then calculate the required stationary probabilities $\{\mathsf{P}_i\}$. In Section 5 we perform simulation to demonstrate the stability/instability of the system depending on whether the conditions on the assignment probabilities are met or violated the mentioned balance equations.

2 Description of the System

In this work we study the extension of the following system with J identical servers and two-priority classes of customers: the first priority customers (class-1) are lost if find all servers busy, while the second priority customers (class-2) join an infinite capacity queue if find all servers busy. Customers of class-i follow Poisson input with rate $\lambda^{(i)}$. The service discipline for class-2 customers is assumed to be FCFS (first-come-first-served). Also we assume that class-i customers have independent identically distributed (iid) service times $\{S_n^{(i)}\}$ with generic element $S^{(i)}$. Denote by $\rho_2 = \lambda^{(2)}\mathsf{E}S^{(2)}$ the traffic intensity of class-2 customers and let P_i be the stationary probability that i servers are occupied by class-1 customers. Such a modified Erlang loss system is motivated and studied in the paper [1] in which in particular the following stability criterion of this system is obtained:

$$\rho_2 + \sum_{i=1}^{J} i\mathsf{P}_i < J. \tag{1}$$

A new distinctive feature of the modified system is that a *multi-type* server assignment for the priority customers is assumed. Each server can serve only a *limited set* of subclasses of class-1 customers and different subclasses of customers in general have different arrival rates. We assume Poisson inputs and server-dependent service rates for the sets of priority subclasses. At the same time, class-2 customers belong to only one type and can be served by any server.

More exactly, we assume J numbered servers $\{1, ..., J\}$ and the first priority (class-1) customers are multi-class divided on subclasses $\{1, ..., I\}$. Subclass customer i can be served by servers from a set $S(i)$, and server j can serve customers from subclasses $C(j)$, for each $i = 1, ..., I$ and $j = 1, ..., J$. Customers of subclass i arrive at rate λ_i, and in general, server j has rate μ_j.

Now we show that condition (1) is also the stability condition of this extended system provided the service rates are the same for all servers. We apply the proof from [1] (see Section 2) and show that relations (20),(21) in [1] are correct for the extended system. Specifically, we show that the completed work (i.e., the total service time) of class-1 customers in interval $[0, t)$ satisfies:

$$\int_0^t \sum_{i=1}^{J} i\mathbf{1}(Q_1(u) = i)du, \ \ t \geq 0, \tag{2}$$

where $\mathbb{1}$ is the indicator function and $Q_1(t)$ is the number of servers occupied by class-1 customers at instant t. Indeed, we can split indicators in (2) into $2^J - 1$ non intersection subsets of indicators:

$$\begin{aligned}
\mathbb{1}(Q_1(t) = 1) &= \sum_{i=1}^{J} \mathbb{1}(\{\text{only the i-th server is busy at instant t}\}),\\
\mathbb{1}(Q_1(t) = 2) &= \sum_{i,j\in\{1,..,J\},\, i\neq j} \mathbb{1}(\{\text{only the i-th and the j-th servers}\\
&\qquad\qquad\qquad\qquad \text{are busy at instant t}\}),\\
&\ldots\\
\mathbb{1}(Q_1(t) = J) &= \mathbb{1}(\{\text{all servers are busy at instant t}\}).
\end{aligned}$$

Then we can obtain the corresponding stationary probabilities as in relations (21) in the paper [1]. We denote by $\hat{V}_1(t)$ the work of class-1 customers accepted by the system in time interval $[0,\, t]$. Note that $\hat{V}_1(t)$ does not include the lost work. Let $W_1(t)$ be the workload (the remaining work to be done) of class-i customers at instant t^-. It is easy to see that,

$$\hat{V}_1(t) = \int_0^t \sum_{i=1}^{J} [i\mathbb{1}(Q_1(u) = i)]du + W_1(t),\ t \geq 0. \tag{3}$$

Because the number of class-1 customers is upper bounded as $Q_1(t) \leq J$, then it is easy to show that the class-1 customer queue is a positive recurrent regenerative process [5] and the following limit exists:

$$\begin{aligned}
\lim_{t\to\infty} \frac{\mathsf{E}\hat{V}_1(t)}{t} &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{i=1}^{J} [i\mathsf{P}(Q_1(u) = i)]du =\\
&= \sum_{i=1}^{J} i \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathsf{P}(Q_1(u) = i)du = \sum_{i=1}^{J} i\mathsf{P}_i.
\end{aligned} \tag{4}$$

The key observation is that the stationary probability P_i, that i servers are occupied by class-1 customers, can be represented as the limiting fraction of the corresponding busy time. This observation implies the last equality in (4). The rest of the proof is the same as in [1]. We note that if the service rates are different then (2) does not express the completed work of class-1 customers, and the equality (3) does not hold. Therefore we further assume that the service rates are the same for all servers.

3 Preliminary Results

Now we present some results from paper [2]. We denotes by $X(t)$ the set of numbers of servers which are *not busy* by class-1 customers at time t. The state space of the process $\{X(t)\}$ is $S \subseteq \{1, ..., J\}$, and is all possible combinations of the numbers of servers. When the system is in state S an arriving class-1 customer of subclass i selects a server $j \in S$ with the probability $\mathsf{P}_{i,j}(S)$. These *assignment probabilities* are the control parameters that we can choose to obtain the stationary distribution of the system. If $i \notin C(j)$ then $\mathsf{P}_{i,j}(S) = 0$. If $S = \{k\}$ and $i \in C(k)$ then $\mathsf{P}_{i,k}(S) = 1$ for all $k \in \{1, ..., J\}$. We note that class-2 customers do not affect the assignment and the service of class-1 customers. Therefore all results obtained in the paper [2] for the loss system with multi-class customers and multi-class servers and briefly presented below are applicable for our extended system.

We assume Poisson input and exponential service times for all of customer subclasses. Under this assumption the process $\{X(t)\}$ is a continuous-time Markov chain. If we have arbitrary service time distribution, we obtain the same stationary distribution. See the proof of this in Proposition 7 in the paper [2]. Now we show that one can choose the assignment probabilities in such a way that one can obtain the stationary distribution of the process $\{X(t)\}$ (as in the paper [2]) and also obtain probabilities P_i in (1).

Now we assume that the process $\{X(t)\}$ is *reversible*. Actually, in the paper [2] this statement is claimed but not shown in details (see Section 3 in [2]), and we check it for our example by conducting simulation in Section 5. Because $\{X(t)\}$ is reversible, then the following detailed balance equations hold:

$$\pi(S)\nu_j(S) = \pi(S \backslash \{j\})\mu_j \qquad \text{for all subset } S \subseteq \{1, ..., J\} \text{ and } j \in S, \tag{5}$$

where $\pi(S)$ is the stationary probabilities that servers S are not busy by class-1 customers, $\nu_j(S)$ is the rate at which server $j \in S$ becomes busy, when the system is in state S. From these equations one can obtain the stationary distribution for $S = \{j_1, ..., j_m\}$, $j_1, ..., j_m \in \{1, ..., J\}$ and $m = 1, ..., J$:

$$\pi(S) = \pi(\emptyset) \frac{\mu_{j_1}}{\nu_{j_1}(\{j_1\})} \frac{\mu_{j_2}}{\nu_{j_2}(\{j_1, j_2\})} \frac{\mu_{j_3}}{\nu_{j_3}(\{j_1, j_2, j_3\})} \cdots \frac{\mu_{j_m}}{\nu_{j_m}(S)}, \tag{6}$$

where $\pi(\emptyset)$ normalizes the sum to 1. Further we denote

$$\pi = \pi(S), \quad \pi^{(k)} = \pi(S \backslash \{k\}), \quad \pi^{(\{j,k\})} = \pi(S \backslash \{j, k\}),$$

$$\nu_j = \nu_j(S) \quad \text{and} \quad \nu_j^{(k)} = \nu_j(S \backslash \{k\})$$

for all $S \subseteq \{1, ..., J\}$ and $j, k \in S$.

Proposition 1. *If detailed balance equations* (5) *hold, then the following recursion holds:*

$$\frac{\nu_j}{\nu_k} = \frac{\nu_j^{(k)}}{\nu_k^{(j)}}. \tag{7}$$

Proof. From the detailed balance equations we have:

$$\pi\nu_j = \pi^{(j)}\mu_j,$$
$$\pi\nu_k = \pi^{(k)}\mu_k.$$

Then we obtain:

$$\pi^{(k)}\nu_j^{(k)} = \pi^{(\{j,k\})}\mu_j, \quad j \neq k,$$
$$\pi^{(j)}\nu_k^{(j)} = \pi^{(\{j,k\})}\mu_k, \quad j \neq k,$$

and finally we obtain:

$$\nu_j^{(k)} = \frac{\pi^{(\{j,k\})}\mu_j}{\pi^{(k)}} = \pi^{(\{j,k\})}\frac{\mu_j\mu_k}{\pi\nu_k}, \tag{8}$$

$$\nu_k^{(j)} = \frac{\pi^{(\{j,k\})}\mu_k}{\pi^{(j)}} = \pi^{(\{j,k\})}\frac{\mu_j\mu_k}{\pi\nu_j}. \tag{9}$$

Now recursion (7) follows from (8) and (9) $\square$.

We denote by $\nu(S)$ the rate at which one of the idle servers becomes busy when the system is in state S. It is clear that the following relations holds for all $S \subseteq \{1, ..., J\}$:

$$\nu(S) = \sum_{j\in S}\nu_j(S) = \sum_{i\in C(S)}\lambda_i. \tag{10}$$

Proposition 2. *The equations* (7) *and* (10) *uniquely determine the values of* $\nu_j(S)$ *by the following recursion for all* $S \subseteq \{1, ..., J\}$ *and* $j \in S$*:*

$$\nu_j(S) = \nu(S) \Bigg/ \left(1 + \sum_{k\in S\setminus\{j\}} \frac{\nu_k^{(j)}}{\nu_j^{(k)}}\right). \tag{11}$$

The proof is by induction on size of S (see Proposition 1 in the paper [2]).

This result shows that the stationary probabilities (6) can be expressed explicitly, using (10) and (11).

Proposition 3. *There exist assignment probabilities* $\mathsf{P}_{i,j}(S)$*, for all* $S \subseteq \{1, ..., J\}$*,* $j \in S$*, and* $i \in C(S)$*, which satisfy the following relations:*

$$\nu_j(S) = \sum_{i\in C(j)}\lambda_i\mathsf{P}_{i,j}(S). \tag{12}$$

The proof of this statement can be found in Propositions 2-6 in the [2]. It is important to note that if we choose the assignment probabilities satisfying (12) then the system will be reversible and have the stationary distribution (6) (see Section 3 in [2]).

To obtain the probabilities in stability condition (1) we should sum up the stationary probabilities (6) as follows:

$$\mathsf{P}_k = \sum_{j_1,...,j_{J-k}\in\{1,...,J\}} \pi(\{j_1, ..., j_{J-k}\}) \qquad \text{for all } k \in \{1, ..., J\}. \tag{13}$$

Substituting this expression in inequality (1), we obtain the stability condition of the extended system in an explicit form.

4 An Example

In this Section, we consider an example of the new system to show in detail how to find stationary and assignment probabilities.

Example 1.

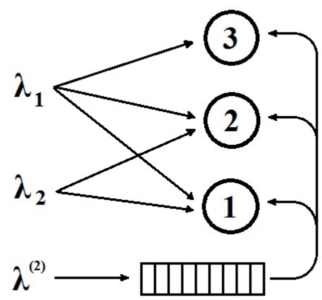

Fig. 1. Example

Let $J = 3$, $I = 2$, $C(1) = \{1, 2\}$, $C(2) = \{1, 2\}$ and $C(3) = \{1\}$ (see Fig.1.). By (6) we have:

$$\pi(\{1\}) = \pi(\emptyset)\frac{\mu_1}{\nu_1(\{1\})}, \quad \pi(\{2\}) = \pi(\emptyset)\frac{\mu_2}{\nu_2(\{2\})}, \quad \pi(\{3\}) = \pi(\emptyset)\frac{\mu_3}{\nu_3(\{3\})},$$
$$\pi(\{1,2\}) = \pi(\emptyset)\frac{\mu_1}{\nu_1(\{1\})}\frac{\mu_2}{\nu_2(\{1,2\})},$$
$$\pi(\{1,3\}) = \pi(\emptyset)\frac{\mu_1}{\nu_1(\{1\})}\frac{\mu_3}{\nu_3(\{1,3\})},$$
$$\pi(\{2,3\}) = \pi(\emptyset)\frac{\mu_2}{\nu_2(\{2\})}\frac{\mu_3}{\nu_3(\{2,3\})},$$
$$\pi(\{1,2,3\}) = \pi(\emptyset)\frac{\mu_1}{\nu_1(\{1\})}\frac{\mu_2}{\nu_2(\{1,2\})}\frac{\mu_3}{\nu_3(\{1,2,3\})}.$$

It is easy to see that:

$$\nu_1(\{1\}) = \lambda_1 + \lambda_2, \quad \nu_2(\{2\}) = \lambda_1 + \lambda_2, \quad \nu_3(\{3\}) = \lambda_1.$$

By relation (11) we have:

$$\nu(\{1,2\}) = \lambda_1 + \lambda_2, \quad \nu(\{2,3\}) = \lambda_1 + \lambda_2, \quad \nu(\{1,3\}) = \lambda_1 + \lambda_2,$$
$$\nu(\{1,2,3\}) = \lambda_1 + \lambda_2.$$

From formula (11) it follows that,

$$\nu_1(\{1,2\}) = \nu(\{1,2\})\left(1+\frac{\nu_2(\{2\})}{\nu_1(\{1\})}\right)^{-1} = \frac{\lambda_1+\lambda_2}{2},$$
$$\nu_1(\{1,3\}) = \nu(\{1,3\})\left(1+\frac{\nu_3(\{3\})}{\nu_1(\{1\})}\right)^{-1} = \frac{(\lambda_1+\lambda_2)^2}{2\lambda_1+\lambda_2}.$$

Similarly, we obtain:

$$\nu_2(\{1,2\}) = \frac{\lambda_1+\lambda_2}{2},$$
$$\nu_2(\{2,3\}) = \frac{(\lambda_1+\lambda_2)^2}{2\lambda_1+\lambda_2},$$
$$\nu_3(\{1,3\}) = \nu_3(\{2,3\}) = \frac{\lambda_1(\lambda_1+\lambda_2)}{2\lambda_1+\lambda_2},$$
$$\nu_1(\{1,2,3\}) = \nu(\{1,2,3\})\left(1+\frac{\nu_2(\{2,3\})}{\nu_1(\{1,3\})}+\frac{\nu_3(\{2,3\})}{\nu_1(\{1,2\})}\right)^{-1} = \frac{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2)}{2(3\lambda_1+\lambda_2)},$$
$$\nu_2(\{1,2,3\}) = \nu(\{1,2,3\})\left(1+\frac{\nu_1(\{1,3\})}{\nu_2(\{2,3\})}+\frac{\nu_3(\{1,3\})}{\nu_2(\{1,2\})}\right)^{-1} = \frac{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2)}{2(3\lambda_1+\lambda_2)},$$
$$\nu_3(\{1,2,3\}) = \nu(\{1,2,3\})\left(1+\frac{\nu_1(\{1,2\})}{\nu_3(\{2,3\})}+\frac{\nu_2(\{1,2\})}{\nu_3(\{1,3\})}\right)^{-1} = \frac{\lambda_1(\lambda_1+\lambda_2)}{3\lambda_1+\lambda_2}.$$

Now we can write down the stationary probabilities $\pi(S)$ by substituting the obtained above rates $\nu_i(S)$ in expression (6):

$$\pi(\{1\}) = \pi(\emptyset)\frac{\mu_1}{\lambda_1+\lambda_2}, \quad \pi(\{2\}) = \pi(\emptyset)\frac{\mu_2}{\lambda_1+\lambda_2}, \quad \pi(\{3\}) = \pi(\emptyset)\frac{\mu_3}{\lambda_1},$$
$$\pi(\{1,2\}) = \pi(\emptyset)\frac{2\mu_1\mu_2}{(\lambda_1+\lambda_2)^2},$$
$$\pi(\{1,3\}) = \pi(\emptyset)\frac{\mu_1\mu_2(2\lambda_1+\lambda_2)}{\lambda_1(\lambda_1+\lambda_2)^2},$$
$$\pi(\{2,3\}) = \pi(\emptyset)\frac{\mu_2\mu_3(2\lambda_1+\lambda_2)}{\lambda_1(\lambda_1+\lambda_2)^2},$$
$$\pi(\{1,2,3\}) = \pi(\emptyset)\frac{2\mu_1\mu_2\mu_3(3\lambda_1+\lambda_2)}{\lambda_1(\lambda_1+\lambda_2)^3}, \tag{14}$$

where, recall, we use normalization condition to find $\pi(\emptyset)$. Now, using (13), it is easy to write down the stationary probabilities P_i:

$$\begin{aligned}
\mathsf{P}_0 &= \pi(\{1,2,3\}),\\
\mathsf{P}_1 &= \pi(\{1,2\}) + \pi(\{2,3\}) + \pi(\{1,3\}),\\
\mathsf{P}_2 &= \pi(\{1\}) + \pi(\{2\}) + \pi(\{3\}),\\
\mathsf{P}_3 &= \pi(\emptyset).
\end{aligned} \tag{15}$$

Finally, from (12), we can obtain the assignment probabilities as follows:

$$\begin{aligned}
\lambda_1 \mathsf{P}_{1,1}(\{1,2\}) + \lambda_2 \mathsf{P}_{2,1}(\{1,2\}) &= \nu_1(\{1,2\}), \\
\lambda_1 \mathsf{P}_{1,1}(\{1,3\}) + \lambda_2 &= \nu_1(\{1,3\}), \\
\lambda_1 \mathsf{P}_{1,1}(\{1,2,3\}) + \lambda_2 \mathsf{P}_{2,1}(\{1,2,3\}) &= \nu_1(\{1,2,3\}), \\
\lambda_1 \mathsf{P}_{1,2}(\{1,2\}) + \lambda_2 \mathsf{P}_{2,2}(\{1,2\}) &= \nu_2(\{1,2\}), \\
\lambda_1 \mathsf{P}_{1,2}(\{2,3\}) + \lambda_2 &= \nu_2(\{2,3\}), \\
\lambda_1 \mathsf{P}_{1,2}(\{1,2,3\}) + \lambda_2 \mathsf{P}_{2,2}(\{1,2,3\}) &= \nu_2(\{1,2,3\}), \\
\lambda_1 \mathsf{P}_{1,3}(\{1,3\}) &= \nu_3(\{1,3\}), \\
\lambda_1 \mathsf{P}_{1,3}(\{2,3\}) &= \nu_3(\{2,3\}), \\
\lambda_1 \mathsf{P}_{1,3}(\{1,2,3\}) &= \nu_3(\{1,2,3\}), \\
\mathsf{P}_{1,1}(\{1,2\}) + \mathsf{P}_{1,2}(\{1,2\}) &= 1, \\
\mathsf{P}_{1,1}(\{1,3\}) + \mathsf{P}_{1,3}(\{1,3\}) &= 1, \\
\mathsf{P}_{1,2}(\{2,3\}) + \mathsf{P}_{1,3}(\{2,3\}) &= 1, \\
\mathsf{P}_{2,1}(\{1,2\}) + \mathsf{P}_{2,2}(\{1,2\}) &= 1, \\
\mathsf{P}_{1,1}(\{1,2,3\}) + \mathsf{P}_{1,2}(\{1,2,3\}) + \mathsf{P}_{1,3}(\{1,2,3\}) &= 1, \\
\mathsf{P}_{2,1}(\{1,2,3\}) + \mathsf{P}_{2,2}(\{1,2,3\}) &= 1.
\end{aligned}$$

Solving this system we obtain:

$$\begin{aligned}
\lambda_1 \mathsf{P}_{1,1}(\{1,2\}) + \lambda_2 \mathsf{P}_{2,1}(\{1,2\}) &= \frac{\lambda_1 + \lambda_2}{2}, \\
\mathsf{P}_{1,1}(\{1,3\}) &= \frac{\lambda_1}{2\lambda_1 + \lambda_2}, \\
\mathsf{P}_{1,3}(\{1,3\}) &= \frac{\lambda_1 + \lambda_2}{2\lambda_1 + \lambda_2}, \\
\mathsf{P}_{1,2}(\{2,3\}) &= \frac{\lambda_1}{2\lambda_1 + \lambda_2}, \\
\mathsf{P}_{1,3}(\{2,3\}) &= \frac{\lambda_1 + \lambda_2}{2\lambda_1 + \lambda_2}, \\
\mathsf{P}_{1,2}(\{1,2\}) &= 1 - \mathsf{P}_{1,1}(\{1,2\}), \\
\mathsf{P}_{2,2}(\{1,2\}) &= 1 - \mathsf{P}_{2,1}(\{1,2\}), \\
\lambda_1 \mathsf{P}_{1,1}(\{1,2,3\}) + \lambda_2 \mathsf{P}_{2,1}(\{1,2,3\}) &= \frac{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2)}{2(3\lambda_1 + \lambda_2)}, \\
\mathsf{P}_{1,3}(\{1,2,3\}) &= \frac{(\lambda_1 + \lambda_2)}{3\lambda_1 + \lambda_2}, \\
\mathsf{P}_{1,2}(\{1,2,3\}) &= \frac{2\lambda_1}{3\lambda_1 + \lambda_2} - \mathsf{P}_{1,1}(\{1,2,3\}), \\
\mathsf{P}_{2,2}(\{1,2,3\}) &= 1 - \mathsf{P}_{2,1}(\{1,2,3\}).
\end{aligned} \tag{16}$$

This system has *infinity number of solutions*, because we can vary some probabilities, for example $\mathsf{P}_{1,1}(\{1,2\})$ and $\mathsf{P}_{1,1}(\{1,2,3\})$. We give some solutions of this system in Section 5 below.

5 Simulation

To check theoretical results presented above, we conduct discrete-event simulation of the system described in Example 1. In addition, we assume that

$$\lambda_1 = \lambda_2 = 10, \quad \mu_1 = \mu_2 = \mu_3 = 10.$$

Now we can calculate theoretical stationary probabilities (14):

$$\begin{gathered}
\pi(\emptyset) = 1/6, \quad \pi(\{1\}) = 1/12, \\
\pi(\{2\}) = 1/12, \quad \pi(\{3\}) = 1/6, \\
\pi(\{1,2\}) = 1/12, \quad \pi(\{1,3\}) = 1/8, \\
\pi(\{2,3\}) = 1/8, \quad \pi(\{1,2,3\}) = 1/6.
\end{gathered} \tag{17}$$

Also we can write down the stationary probabilities P_i from (14) and (15)

$$\mathsf{P}_0 = 1/6, \quad \mathsf{P}_1 = 1/3, \quad \mathsf{P}_2 = 1/3, \quad \mathsf{P}_3 = 1/6. \tag{18}$$

$S=\{1,2\}$				$S=\{1,3\}$		$S=\{2,3\}$		$S=\{1,2,3\}$					
$\mathsf{P}_{1,1}$	$\mathsf{P}_{1,2}$	$\mathsf{P}_{2,1}$	$\mathsf{P}_{2,2}$	$\mathsf{P}_{1,1}$	$\mathsf{P}_{1,3}$	$\mathsf{P}_{1,2}$	$\mathsf{P}_{1,3}$	$\mathsf{P}_{1,1}$	$\mathsf{P}_{1,2}$	$\mathsf{P}_{2,1}$	$\mathsf{P}_{2,2}$	$\mathsf{P}_{1,3}$	$\boldsymbol{\delta(t)}$
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	$\frac{1}{3}$	$\frac{1}{3}$	0.5	0.5	$\frac{1}{3}$	**0.0322**
0.5	0.5	0.5	0.5	0.0	1.0	0.0	1.0	$\frac{1}{6}$	$\frac{1}{6}$	0.5	0.5	$\frac{2}{3}$	**0.0459**
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	**0.0870**
0.0	1.0	0.0	1.0	1.0	0.0	1.0	0.0	0.0	1.0	0.0	1.0	0.0	**0.1559**

Table 1. The assignment probabilities violating equation system (16): row 1 is uniform distribution; row 2 is uniform servers workload; row 3 is high workload on server 2; row 4 is maximum workload on server 2.

To compare these probabilities with simulation results, we use the Euclidean distance between two probabilities distribution:

$$\delta(t) = \sqrt{\sum_{S \subseteq \{1,\dots,J\}} (\pi(S) - \pi^*(S,t))^2},$$

where

$$\pi^*(S,t) = \frac{T(S,t)}{t}, \ \ t > 0,$$

$S=\{1,2\}$				$S=\{1,3\}$		$S=\{2,3\}$		$S=\{1,2,3\}$					
$\mathbf{P_{1,1}}$	$\mathsf{P}_{1,2}$	$\mathsf{P}_{2,1}$	$\mathsf{P}_{2,2}$	$\mathsf{P}_{1,1}$	$\mathsf{P}_{1,3}$	$\mathsf{P}_{1,2}$	$\mathsf{P}_{1,3}$	$\mathbf{P_{1,1}}$	$\mathsf{P}_{1,2}$	$\mathsf{P}_{2,1}$	$\mathsf{P}_{2,2}$	$\mathsf{P}_{1,3}$	$\boldsymbol{\delta(t)}$
0.1	0.9	0.9	0.1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.1**	0.4	0.65	0.35	0.5	**0.0016**
0.1	0.9	0.9	0.1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.3**	0.2	0.45	0.55	0.5	**0.0013**
0.1	0.9	0.9	0.1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.5**	0.0	0.25	0.75	0.5	**0.0010**
0.5	0.5	0.5	0.5	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.1**	0.4	0.65	0.35	0.5	**0.0009**
0.5	0.5	0.5	0.5	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.3**	0.2	0.45	0.55	0.5	**0.0012**
0.5	0.5	0.5	0.5	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.5**	0.0	0.25	0.75	0.5	**0.0009**
0.9	0.1	0.1	0.9	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.1**	0.4	0.65	0.35	0.5	**0.0013**
0.9	0.1	0.1	0.9	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.3**	0.2	0.45	0.55	0.5	**0.0012**
0.9	0.1	0.1	0.9	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	**0.5**	0.0	0.25	0.75	0.5	**0.0015**

Table 2. Cases of assignment probabilities that satisfy equation system (16): we can vary only 2 marked probabilities; $\delta(t)$ is much less than in Table 1.

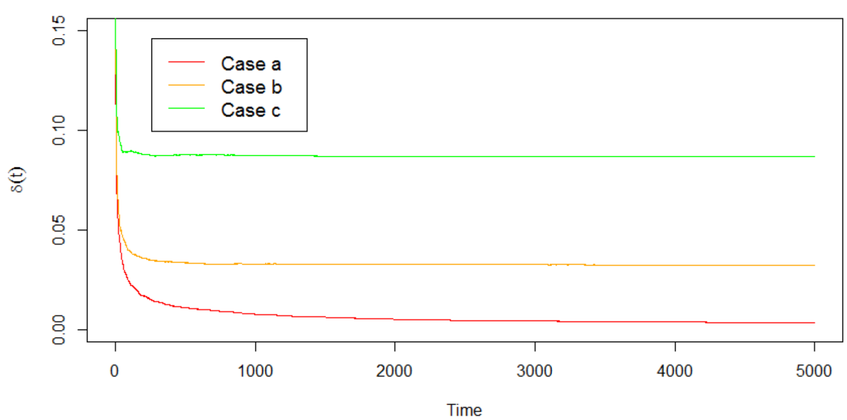

Fig. 2. Difference between theoretical and simulated stationary probabilities (Sample mean $\delta(t)$): case a) corresponds to row 3 in Table 1; case b) corresponds to row 1 in Table 1; case c) corresponds to row 1 in Table 2.

and $T(S,t)$ is the time, in interval $[0,t)$, when the system is in the state S. Also we introduce $l(t)$ (loss rate), the number of customer losses per time unit in interval $[0,t)$.

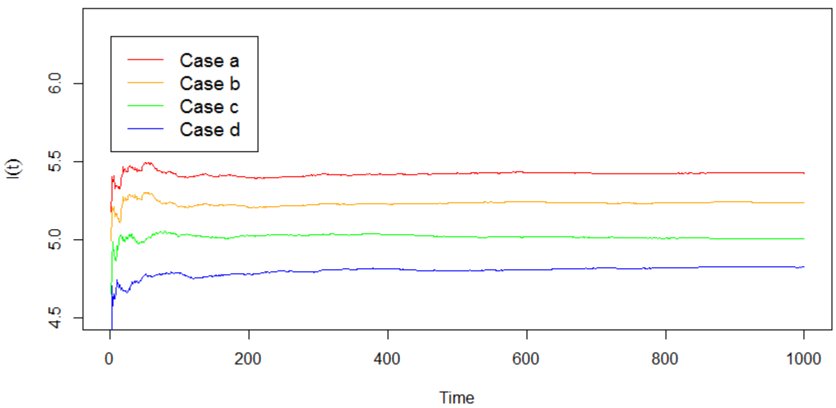

Fig. 3. Number of losses per time unit in interval $[0, t)$ (Sample mean $l(t)$): case a) corresponds to row 4 in Table 1; case b) corresponds to row 3 in Table 1; case c) corresponds to row 1 in Table 2; case d) corresponds to row 2 in Table 1.

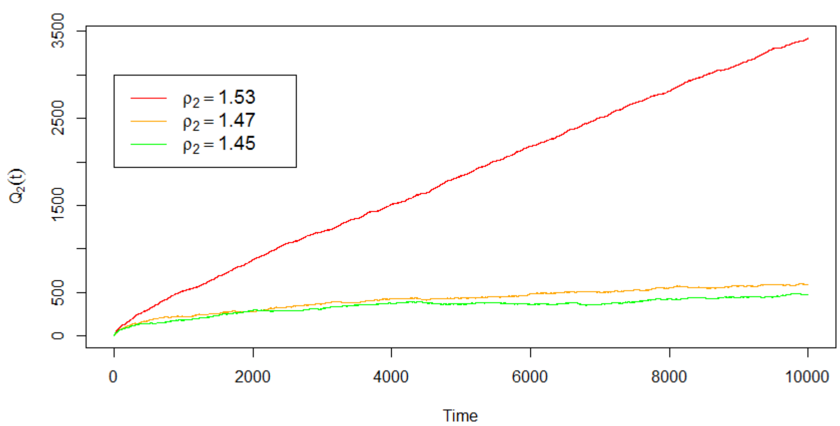

Fig. 4. Sample mean queue size of class-2 customers for Pareto service times for case row 1 in Table 2. Stability condition (1) is $\rho_2 < 1.50$.

We obtain the sample mean based on 100 paths of $\delta(t)$ and present their stationary values in Tables 1 and 2. We consider some different sets of assignment probabilities: 4 cases violate equation system (16) (see Table 1) and 9 cases satisfy it (see Table 2). To satisfy equation system (16) we can vary $\mathsf{P}_{1,1}(\{1,2\})$ and $\mathsf{P}_{1,1}(\{1,2,3\})$ (free variables, marked columns) while other probabilities depend on free variables or are uniquely determined. It is easy to see that $\delta(t)$ in Table 1 is much larger than in Table 2, that confirms that stationary probabilities (17) are found correctly (for cases in Table 2) and the process is reversible. One can see on Fig.2 that $\delta(t)$ converges for both in the cases when the system (16) is satisfied and in the cases when it is violated. We also show that number of customer losses $l(t)$ also converges but for the cases when the system (16) is violated $l(t)$ is not much larger than for the cases when the system (16) is satisfied (see Fig.3). Moreover, case d) shows that we can obtain $l(t)$ less than for the cases when the system (16) is satisfied. This example shows that product form distribution is not optimal to minimize losses.

Finally, we calculate the stability condition (1) and, provided this condition is satisfied, check stability by simulation queue size of class-2 customers for Pareto service times. In our example the number of servers $J = 3$, and from (18) and (1) we obtain the upper bound for ρ_2 as

$$\rho_2 < 1.5. \tag{19}$$

In the experiments, we construct the sample mean queue size $Q_2(t)$ based on 300 paths queue size of class-2 customers for different values of ρ_2 (see Fig.4). It is easy to see that, if (19) does not hold, then $Q_2(t)$ increases linearly to infinity reflecting *strong instability*. On the other hand, when condition (19) is satisfied then we see that all paths are stable.

6 Conclusion

We consider a modified Erlang loss system with multi-class multi-server first priority customers and a queue for second priority customers. In this work we show that the stationary distribution describing this system has the product form and is the same as in the system considered in the paper [2]. Furthermore, we use these stationary probabilities to calculate the stability condition of the entire system expressed as the upper bound for the traffic intensity generated by the second priority customers. Simulation illustrates that the assignment probabilities, satisfying conditions in the considered example, indeed imply the product form of the stationary distribution. Moreover, we show that the stability condition holds for the example studied in Section 4. On the other hand, simulation also shows that the product form distribution setting is not optimal to minimize the losses in the system.

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References

1. Morozov, E., Rogozin, S., Nguyen, H.Q., Phung-Duc, T.: Modified Erlang loss system for cognitive wireless networks. Journal of Mathematical Sciences (2020) (submitted).
2. Adan, I., Hurkens, C., Weiss, G.: A reversible erlang loss system with multitype customers and multitype servers. Probability in the Engineering and Informational Sciences 24(4), 535 - 548 (2010).

3. Akutsu, K. and Phung-Duc, T.: Analysis of retrial queues for cognitive wireless networks with sensing time of secondary users, Lecture Notes in Computer Science, LNCS 11688, pp. 77-91 (2019).
4. Akyildiz, I. F., Lee, W.-Y., Vuran, M. C., Mohanty, S.: NeXt generation dynamic spectrum access cognitive radio wireless networks: a survey. Computer Networks 50(13), 2127–2159 (2006).
5. Asmussen, S. Applied probability and Queues. 2nd edn. Springer, Springer-Verlag New York (2003).
6. Konishi, Y., Masuyama, H., Kasahara, S., Takahashi, Y.: Performance analysis of dynamic spectrum handoff scheme with variable bandwidth demand of secondary users for cognitive radio networks. Wireless Networks 19, 607–617 (2013).
7. Salameh, O., De Turck, K., Bruneel, H., Blondia, C., Wittevrongel, S.: Analysis of secondary user performance in cognitive radio networks with reactive spectrum handoff. Telecommunication Systems 65, 539–550 (2017).
8. Wong, E. W. A., Foh, C. H.: Analysis of cognitive radio spectrum access with finite user population. IEEE Communications Letters 13(5), 294–296 (2009).
9. Down, D. G., Lewis, M. E.: Dynamic load balancing in parallel queueing systems: stability and optimal control. European Journal of Operational Research, 168, 509–519 (2006).
10. Tsai, Y. C., Argon, N. T.: Dynamic server assignment policies for assembly-type queues with flexible servers. Naval Research Logistics, 55(3), 234–251 (2008).
11. Agnihothri, S. R., Mishra, A. K., Simmons, D. E.: Workforce cross-training decisions in field service systems with two job types. The Journal of the Operational Research Society, 54(4), 410–418 (2003).
12. Ahghari, M., Balcioglu, B.: Benefits of cross-training in a skill-based routing contact center with priority queues and impatient customers. IIE Transactions, 41, 524–536 (2009).
13. Tekin, E., Hopp, W. J., Van Oyen, M. P.: Pooling strategies for call center agent cross-training. IIE Transactions, 41(6), 546–561 (2009).
14. Ahn, H.-S., Duenyas, I., Zhang, Q. R.: Optimal control of a flexible server. Advances in Applied Probability, 36, 139–170 (2004).
15. Stolyar, A. L., Tezcan, T.: Control of systems with flexible multi-server pools: a shadow routing approach. Queueing Systems, 66, 1–51 (2010).