Quorum Analysis for the "Pirate Game" Problem*

Ilya Chernov[1] and Evgeny Ivashko[1,2]

[1] Institute of Applied Mathematical Research, Karelian Research Center of the Russian Academy of Sciences, Petrozavodsk, Russia
[2] Petrozavodsk State University
{chernov,ivashko}@krc.karelia.ru
http://mathem.krc.karelia.ru/

Abstract. The "Pirate game" is a popular mathematical puzzle, the problem aimed at demonstration of backward induction. It is also a shining example of non-cooperative behaviour of rational agents in mathematical game theory. A linearly ordered crew votes for the captain's sharing offer. The captain needs the approval by a given fraction of the crew called quorum at the minimal cost. In this paper, we present the comprehensive quorum analysis on strategies and payoffs for the "Pirate game" problem, for any crew size and any quorum.

Keywords: Pirate Game · Puzzle for Pirates · Quorum Analysis · Non-Cooperative Game Theory

1 Introduction and Related Work

The "Pirate game" (also known as the "Puzzle for pirates") is a well-known widely-used mathematical puzzle, which often stirs up disputes[3]. The first appearing of the problem in scientific literature seems to be in the book by E. Moulin [6] as an example; we found no earlier publications of this problem. The "Pirate game" is the following[4]:

> There are five rational pirates (in strict order of seniority A, B, C, D and E) who found 100 gold coins. They must decide how to distribute them. The pirate world's rules of distribution say that the most senior pirate first proposes a plan of distribution. The pirates, including the proposer, then vote on whether to accept this distribution. If the majority accepts the plan, the coins are dispersed and the game ends. In case of a tie vote, the proposer has the casting vote. If the majority rejects the plan, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again. The process repeats until a plan is accepted or if there is one pirate left.

* Supported by Russian Foundation for Basic Research, project 18-07-00628
[3] http://www.mytechinterviews.com/5-pirates-fight-for-100-gold-coins
[4] Cited from https://en.wikipedia.org/wiki/Pirate_game

Copyright © 2020 for this paper by its authors. Use permitted underCre

CEUR-WS.org/Vol-2792/paper2.pdf

> Pirates base their decisions on four factors. First of all, each pirate wants to survive. Second, given survival, each pirate wants to maximize the number of gold coins he receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal. And finally, the pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate.

Using backward induction, one can find the final solution: A: 98 coins; B: 0 coins; C: 1 coin; D: 0 coins; E: 1 coin.

In [6], there is no restriction on the number of pirates (N) and the general solution is the following:

> For $N = 2p + 1$ and $N = 2p + 2$, the most senior pirate gets $(100 - p)$ coins. One coin gets each of p pirates whose numbers have the same parity as the number of the most senior pirate.

Obviously, this solution works while the number of coins is more than half of the number of pirates ($N > 2M$). An extension of the "Pirate game" to the case of a larger number of pirates is presented in [7] by I. Stewart.

In popular scientific literature, there are also a few other extensions; for example, when the needed quorum is greater than 0.5[5] or "six pirates and a single coin"[6].

A similar to the "Pirate game" problem is considered in [4], but there the junior crew member makes an offer to be approved by everybody ($Q = 1$) or everybody except at most one. Also, there are several approve-sharing problems with envious and/or greedy same-rank pirates, where also the share must be approved by every crew member (e.g, [2]).

Work [3] analyses the "Pirate Game" by a game-theoretic solution concept of "subgame perfect strong Nash equilibrium," as the subgame version of the strong Nash equilibrium. In particular, the thesis provides an analysis of the general form of the "Pirate Game", by determining the maximum number of coins that the most senior pirate could guarantee.

Paper [5] develops a quantum model for the voting system in the "Pirate game".

Most of the discussed problem variations (with rare exceptions) are based on quorum $Q = 0.5$, i.e., the proposition to be accepted needs at least 50% of votes. However, it is still an open question, how the quorum size affects on payoffs and strategies. This paper aims at making the comprehensive quorum analysis (with $0 \leq Q \leq 1$) of the "Pirate Game". Various generalizations of this game will be the subject of our future work.

The paper is organized as follows. Section 2 describes the formal problem. Section 3 provides the quorum analysis with subsections of different solutions: small quorums ($0 \leq Q \leq 0.5$), the ranges $0.5 < Q \leq 0.6$, $0.6 < Q \leq 0.625$

[5] https://en.wikibooks.org/wiki/Puzzles/Logic_puzzles/Gold_Coins
[6] http://www.mytechinterviews.com/6-pirates-fight-for-1-gold-coin

and $0.625 < Q \leq 2/3$, and high quorums $2/3 < Q \leq 1$. Section 4 provides possible applications of the problem. Finally, section 5 gives the conclusion and final remarks.

2 The Problem

We consider the problem of sharing with voting in the form presented in [6] with some generalization. The context of pirates is convenient, because no trust, neither cooperative behaviour, between the players are suggested.

The problem is as follows. A crew of N pirates is linearly ordered by seniority. The senior pirate is called the captain. The captain offers to share a number M of coins; we assume that this number is sufficiently large. The crew votes for or against the offer; if the number of the votes for the offer is at least QN, where $0 \leq Q \leq 1$ is called the quorum, the offer is accepted. Otherwise, the captain is expelled from the crew and does not take part in the subsequent voting. The new captain offers a new sharing.

Let the junior member of the crew have number one; the higher the number, the higher the position in the hierarchy. Crew members vote rationally and in their interests, payments are discrete and are multiples of one coin. A member of the crew who is offered a payout of all M coins votes for the share.

The number of M coins is sufficiently large in the following sense: if the sums offered to the members of the crew do not depend on M, then the captain gets more than anyone of the crew, after paying the offered sum to every member.

In a situation of indifference, that is, if a crew member is offered the same share that he expects to receive from the new captain, he can vote either for, or against the offer. The captain, therefore, needs to assume that all such crew members may vote against the offer.

3 The Quorum Analysis

In this paper, we solve the problem for all values $0 \leq Q \leq 1$. Besides the payoff functions for any crew size N and any quorum Q, we would like to emphasize the following results:

- In a general case, the captain has some freedom in deciding who is offered some money and who is not. This is not the case for $Q = 0.5$.
- The quorum span $[0, 1)$ is separated into intervals, the payoff for a given N is the same within the interval; also, the asymptotic average individual payoff grows piecewise-linearly with respect to Q and may jump at intervals' ends.
- The span $[0, 2/3)$ is divided into just five intervals, while the rest consists of infinitely many intervals that condense near 1.

Note that the rules of the voting can be different: instead of "at least" Q, it can be "more than" Q. This is also the case when the captain does not vote. However, one problem is equivalent to the other replacing Q by $Q + \delta$ with sufficiently small δ; in fact, it is sufficient to choose $\delta < 1/N$.

Below we consider the following cases: $0 \leq Q \leq 0.5$ and $Q = 0.5 + \delta$, which are different. Then we show, that this pattern is valid for $0.5 < Q \leq 0.6$, while a similar pattern holds for $0.6 < Q \leq 0.625$. After that, we consider the case $Q = 2/3$ and show that it is applicable also for $0.625 < Q \leq 2/3$. Finally, we consider completely different case $2/3 \leq Q \leq 1$.

3.1 Small Quorums: $0 \leq Q \leq 0.5$

The classical problem for $Q = 0.5$ has the well-known solution presented in [6] that allows the captain to keep the most of the sum for himself. It is proved by induction. The payoff (to every crew member except the captain) is either one coin or nothing depending on the parity of the member's rank.

Indeed, for $N = 2$ the captain can keep everything for himself, being himself one half of the team. For $N = 3$, he needs one more vote, so he offers one coin to the junior member, who otherwise would get nothing.

So, the payout scheme is

$$C\,\underbrace{(01)(01)\dots(01)}_{n-1\text{ times}}(0) \quad \text{for} \quad N = 2n,$$
$$C\,\underbrace{(01)(01)\dots(01)}_{n\text{ times}} \quad \text{for} \quad N = 2n+1,$$

Here the payoffs of the crew members are shown from the senior (left) to the junior (right), with the captain (who takes the rest) denoted by C. Parentheses are used for visual grouping only.

The captain can use the same payout scheme for quorums less than 0.5:

$$C\,\underbrace{(0?)(0?)\dots(0?)}_{n-1\text{ times}}(0) \quad \text{for} \quad N = 2n, \tag{1}$$
$$C\,\underbrace{(0?)(0?)\dots(0?)}_{n\text{ times}} \quad \text{for} \quad N = 2n+1, \tag{2}$$

arbitrarily choosing the necessary number of people from the group marked with question marks. For $Q = 0.5$ all these members are offered one coin, so question marks can be replaced by ones. The mark 0 (or 1) means the offer, the question mark means that this member might be chosen for a one-coin offer, but the captain needs fewer members than the total amount of this category (or exactly this number). However, this scheme is optimal for $Q > 1/3$.

For lower values, i.e., $0 < Q \leq 1/3$, the captain offers nothing to every crew member for any $N \leq 1/Q$, because his own vote is enough. Then he offers nothing to number 2 and has at least two members to choose from for one more vote. For the next N, these junior members are not reliable, but number 3 would agree for one coin. After that, the parity pattern continues. Let us illustrate this for $Q = 0.25$, starting from $N = 5$:

$$\begin{aligned}
&C0???, \quad 1 \text{ to choose},\\
&C0?000, \quad 1 \text{ to choose}\\
&C0?0???, \quad 1 \text{ to choose},\\
&C0?0?000, \quad 1 \text{ to choose},\\
&C0?0?0???, \quad 2 \text{ to choose}.
\end{aligned}$$

So, the pattern is as follows: for $N = 2n$, $n > 2$, several juniors get nothing, as well as every crew member with even number; other can hope for one coin. For $N = 2n + 1$, $n \geq 2$, the same junior members and those with odd numbers may hope for one coin, other are offered nothing.

The number of these junior members, denote it by J, is the minimal J that satisfies the condition $J > 1/Q - 2$, because $J + 2$ is the first N when the captain needs one more vote besides his own. So, in the general case the pattern is the same, only $N \geq J + 2$ (i.e., $N > Q^{-1} - 2$):

$$C\,\underbrace{(0?)(0?)\dots(0?)}_{l-1 \text{ times}}(0)\underbrace{?\dots?}_{J} \quad \text{for} \quad N - J = 2l,\ l \geq 1, \tag{3}$$

$$C\,\underbrace{(0?)(0?)\dots(0?)}_{l \text{ times}}\underbrace{0\dots0}_{J} \quad \text{for} \quad N - J = 2l + 1,\ l \geq 1, \tag{4}$$

The special case $Q = 0$ is trivial: the captain takes everything for any crew size N.

3.2 The Range $0.5 < Q \leq 0.6$

Let us consider the case when the captain needs the approval of more than one half of the crew ($Q = 0.5 + \delta$); alternatively, this is the case when the captain does not have a vote.

The distribution of payments by players depends on the remainder of dividing N by 3. Players are divided into three types: deprived, who are offered nothing; privileged, who are offered one coin; and the others. The captain chooses the necessary amount from this third category and offers them two coins; others are offered nothing. This third category is marked by the question mark.

$$C\,\underbrace{(01?)(01?)\dots(01?)}_{n-1 \text{ times}}10 \text{ for } N = 3n, \tag{5}$$

$$C\,\underbrace{(01?)(01?)\dots(01?)}_{n-1 \text{ times}}0?1 \text{ for } N = 3n + 1, \tag{6}$$

$$C\,\underbrace{(01?)(01?)\dots(01?)}_{n-1 \text{ times}}010? \text{ for } N = 3n + 2. \tag{7}$$

The base of induction is checked directly: for $N = 2$, the junior member of the crew takes everything; here we use the assumption that if a member is offered everything, he votes for the offer. Therefore, for $N = 3$, the captain simply offers the second member one coin, which gives the case $N = 3n$ for $n = 1$.

With $N = 4$, the captain offers one coin to the youngest, but he needs another vote, which can only be number 2, who would be offered one coin by the next captain. Therefore, to guarantee his vote, the captain offers him two coins. Formally, the captain selects one member from the third category, which has a single member; this is $N = 3n + 1$ for $n = 1$.

With $N = 5$, the captain offers one coin to number 3, which would be offered nothing by the next captain; one more vote is needed. Number 2 wants to become a captain and get everything, so he is not offered anything, and there remains a choice of two crew members (1 and 2). Number 2 would be offered two coins by the new captain and may not agree to two; therefore, it is more reliable to offer two coins to the youngest (he falls into the third category). So, we have the case $N = 3n + 2$ for $n = 1$.

The induction step is now obvious. Every one offered nothing on the previous step are offered one; this gives the captain n votes. Nothing is offered to those who could hope for two coins, because they may refuse even the offer of two. There remain at least $n - 1$ crew members, who would be offered one coin on the previous step. Among them, the captain needs to choose enough people to get the quorum and offer them two coins.

Thus, the series of (?10) transforms to (0?1) (the cyclic shift to the right), the same is true for the series (0?1) and the series (10?). The series (01?) becomes (1?0) (the left shift). Then the payout at $N = 3n$ becomes, taking into account the zero payout to number 2,

$$C0\,\underbrace{(1?0)(1?0)\ldots(1?0)}_{n-1 \text{ times}}?1,$$

which, after rearranging the parentheses, gives a payout for $N = 3n + 1$.

The transition to $N = 3n + 2$ is considered in the same way. Now let us consider the transition from $N = 3n + 2$ to $N = 3(n + 1)$. Similarly, we get a series

$$C0\,\underbrace{(1?0)(1?0)\ldots(1?0)}_{n-1 \text{ times}}\,1?10 = C\,\underbrace{(01?)(01?)\ldots(01?)}_{n \text{ times}}\,10,$$

which coincides with the formula for $N = 3(n + 1)$.

The minimum payout follows from the fact that no member of the crew can reduce the payout without guaranteeing his vote.

The proof is complete.

The payoff pattern for $Q = 0.5 + \delta$ is valid up to some δ, because there are more people from the third category, to choose from to make up the quorum. Let us show that this pattern is preserved up to $Q = 0.6$.

The captain, as we have already established, has n votes of those who would be offered nothing by the next captain, his own vote, and has at least $n - 1$

people to choose the necessary number and offer them two coins. So, the captain can to get $2n$ votes out of not more than $3n + 2$.

Moreover, if N is not divisible by 3, then the number of people to choose from is $n + 1$, so the captain can get up to $2n + 1$ votes.

Thus, the captain can get exactly $2/3$ of votes at $N = 3n$, or

$$\frac{2n+1}{3n+1} \text{ for } N = 3n+1, \quad \frac{2n+1}{3n+2} \text{ for } N = 3n+2.$$

The first of these expressions decreases in n and exceeds $2/3$ for all $n \geq 0$, while the second one increases and is equal to $3/5$ at $n = 1$ (so for larger n it is even more).

Due to this equality, it is impossible to use the same pattern for higher Q; let us construct the pattern for this case.

3.3 The range $0.6 < Q \leq 0.625$

For $Q = 0.6 + \delta$ up to $Q = 0.625$ we have the following first steps:

$$\begin{aligned} N = 3: &\quad C10 \\ N = 4: &\quad C021 \\ N = 5: &\quad C\mathit{0132} \\ N = 6: &\quad C(012)03 \\ N = 7: &\quad C(012)310 \end{aligned}$$

After that we have the pattern:

$$C\underbrace{(01?)(01?)\dots(01?)}_{n-1\text{ times}}00?1 \text{ for } N = 3n+2,\ n \geq 2 \tag{8}$$

$$C\underbrace{(01?)(01?)\dots(01?)}_{n-2\text{ times}}0110? \text{ for } N = 3n,\ n \geq 3 \tag{9}$$

$$C\underbrace{(01?)(01?)\dots(01?)}_{n-1\text{ times}}?10 \text{ for } N = 3n+1,\ n \geq 3. \tag{10}$$

This easily checked by induction. This pattern cannot be used for $Q > 0.625$ because for $N = 5$ (emphasized by *italic* font) the captain would lack a zero. For this case, the pattern would depend on divisibility on four.

3.4 The Case $0.625 < Q \leq 2/3$

Consider the situation when the proposal of the captain is accepted if at least two-thirds of the crew (including the captain) vote for it ($Q = 2/3$). The payout scheme depends on divisibility by 4. Players are divided into four types: those, who are offered nothing; those, who are offered one coin, two coins, and the rest: each of the rest may be offered three coins, or nothing, depending on the captain's arbitrary choice. This category is marked by question marks:

$$C\underbrace{(012?)(012?)\dots(012?)}_{n-1\text{ times}}(021) \quad \text{for} \quad N = 4n, \tag{11}$$

$$C\underbrace{(012?)(012?)\dots(012?)}_{n-1\text{ times}}(01?2) \quad \text{for} \quad N = 4n+1, \tag{12}$$

$$C\underbrace{(012?)(012?)\dots(012?)}_{n-1\text{ times}}(0120?) \quad \text{for} \quad N = 4n+2, \tag{13}$$

$$C\underbrace{(012?)(012?)\dots(012?)}_{n\text{ times}}(10) \quad \text{for} \quad N = 4n+3. \tag{14}$$

The induction base is constructed in the same way as in the previous case: for $N = 2$ the captain gives everything to the youngest, providing 100%, and for $N = 3$ he gives one coin to number 2, getting exactly two thirds (payout is $C10$). With $N = 4$, the captain also receives three quarters (payout is $C021$). However, with $N = 5$, two votes are now not sufficient: the third vote is needed. So, the captain must offer three coins to someone who would be offered two by the next captain: the payout is $C0132 = C01?2$). Finally, with $N = 6$, the captain also needs three votes, and he can offer nothing to the one who would be given three coins by the next captain: $C01203 = C0120?$.

By induction, the captain offers one coin to those who would receive nothing (at least n people) and two coins to everyone, who would be offered one (also at least n people). Also, there are at least $n-1$ people who would be offered two coins: the captain chooses the necessary number of people to get the quorum and offers them three coins each, to guarantee their votes. It is easy to check that this algorithm indeed produce the payout scheme for $4n+1$ from $4n$, $4n+2$ from $4n+1$, $4n+3$ from $4n+2$, and $4(n+1)$ from $4n+3$.

So, we can use this pattern also for $Q < 2/3$, only fewer people need to be chosen to secure the quorum. The lower bound for this pattern is $Q = 0.625$ because for this value a better scheme is available. Moreover, with $N = 5$, that is, with $N = 4n+1$ with $n = 1$, the captain is forced to provide the votes of all subordinates, except one (which is number 2, who will not be satisfied with any payment). Since in the previous step two coins would be paid, it is not possible to manage with a smaller amount.

3.5 The High Quorum: $2/3 < Q \leq 1$

Finally, let us consider the quorum $Q > 2/3$.

To solve the problem, we need to strengthen the loyalty assumption: now, if the captain gives everything to one subordinate, then the rest of the crew in a position of indifference are loyal. This is necessary, because already in the case of 3 members the captain needs to offer everything to the junior, while the third member (number 2) gets nothing but still votes for the captain. Otherwise, the captain cannot win.

Under this assumption and for $Q = 1$, the best captain can do is to survive by offering everything to the junior crew member; let us study the case $Q < 1$.

Firstly, as N grows, the captain gives everything to the youngest, until N reaches $\bar{N}_1 = \lceil 1/(1-Q) \rceil$; after that one member can be offered nothing, and this would be the junior. The remaining members (let us call them aristocrats, their number is $A = \lceil (1-Q)^{-1} \rceil - 2$) get one coin, and the rest is taken by the captain.

On the next step, nothing is offered to number 2, who now wishes to become the captain. The aristocrats get two coins, and the junior gets one.

Next, we have an increasing series of 0, 1, 2, ..., $i-1$, then $i+1$, ..., $i+1$, and finally i, (for $i \geq 1$), until $N = \lceil (1-Q)^{-1} \rceil - 2 + i + 1$ exceeds the value

$$\bar{N}_2 = \left\lceil \frac{1}{1-Q} \left(\left\lceil \frac{1}{1-Q} \right\rceil - 1 \right) \right\rceil .$$

At the same time, the captain gets the opportunity to offer nothing to two people, then three, and so on, to be selected among the aristocrats, who receive $i+1$ coins.

The number of aristocrats is constant and equal to A. When $A+1$ pirates can be offered nothing, the captain offers them (and number 2) nothing and they know that. So, at the next step, they can not hope for anything and, therefore, would agree to one coin. The captain gets the opportunity to offer nothing to those who desire more than others, and so on. At the same time, payments to aristocrats grow and they can be "repressed" again.

Let consider the case $Q = 2/3 + \delta$ first. At the beginning up to $N = 3$, the captain has to give everything to the junior. Then the payment has the form

$$\begin{aligned} N = 4: &\quad C110 \\ N = 5: &\quad C0221 \\ N = 6: &\quad C01332 \end{aligned}$$

Next, one more member can be offered nothing, but not both aristocrats. They have hopes and, therefore, it is impossible to rely on their loyalty. Choosing any aristocrat and giving him nothing, we continue:

$$\begin{aligned} N = 7: &\quad C012443 \\ N = 8: &\quad C0123554 \\ N = 9: &\quad C01234665 \end{aligned}$$

Then three men can be offered nothing: number 2 and both aristocrats; we get

$$\begin{aligned} N = 10: &\quad C\mathit{012345006} \\ N = 11: &\quad C\mathit{0123450110} \\ N = 12: &\quad C\mathit{01234001221} \end{aligned}$$

Here *italic* font means that all zeros are used, meaning that the captain has no choice.

At the next step, one more pirate can be offered nothing. They are those who want more and some of those who want three coins (the captain has a choice):

$$N = 13: \quad C012300112332$$

Continuing further, we get

$$\begin{aligned} N = 14: &\quad C\mathit{0123011223003} \\ N = 15: &\quad C\mathit{01230122330110} \\ N = 16: &\quad C012301233001221 \\ N = 17: &\quad C0123012300112332 \\ N = 18: &\quad C\mathit{012301230112230003} \end{aligned}$$

This can be written as a pattern:

$$\begin{aligned} &C\underbrace{(0123)\dots(0123)}_{n-2}(00112332), \quad N = 4n+1, \quad n \geq 3; \\ &C\underbrace{(0123)\dots(0123)}_{n-2}(011223003), \quad N = 4n+2, \quad n \geq 3; \\ &C\underbrace{(0123)\dots(0123)}_{n-2}(0122330110), \quad N = 4n+3, \quad n \geq 3; \\ &C\underbrace{(0123)\dots(0123)}_{n-2}(3001221), \quad N = 4n, \quad n \geq 4; \end{aligned}$$

This scheme is easily checked by induction, provided that the number of those who can be offered nothing, is high enough. The base of induction has been obtained above.

Note that the payout does not exceed three coins, exactly as for the case $Q > 0.625$, provided that the number of crew members is high enough. However, for a small crew, this payoff can be as large as six coins, and for very small everything is offered to the junior.

Also note that for large N the fraction of those who are offered zero are asymptotically $1/4$, though the amount of zeros is asymptotically $1/3$. Therefore, the pattern possesses some stability for large N: the captain has some resources to control the growth of the individual payoff, provided that the crew is large enough.

The pattern remains valid for positive δ up to some threshold value; then it is replaced by a similar pattern. The threshold value is 0.7: this can be checked by considering the patterns in *italics*. There the captain has enough zeros, in particular at $N = 10$, where he needs exactly 7 votes: higher Q would demand 8, so the pattern would be destroyed.

The pattern for $Q = 0.7 + \delta$ is constructed similarly, the first steps (up to $N = 9$) are the same. Then:

$$
\begin{aligned}
N = 10 :& \quad C012345776 \\
N = 11 :& \quad C\mathit{0123456007} \\
N = 12 :& \quad C\mathit{01234560110} \\
N = 13 :& \quad C\mathit{012345001221}
\end{aligned}
$$

After that, for $N \geq 14$, we have the pattern:

$$C\,\underbrace{(0123)\ldots(0123)}_{n-2}(000112332), \quad N = 4n+2, \quad n \geq 3;$$

$$C\,\underbrace{(0123)\ldots(0123)}_{n-2}(0111223003), \quad N = 4n+3, \quad n \geq 3;$$

$$C\,\underbrace{(0123)\ldots(0123)}_{n-3}(01222330110), \quad N = 4n, \quad n \geq 4;$$

$$C\,\underbrace{(0123)\ldots(0123)}_{n-2}(33001221), \quad N = 4n+1, \quad n \geq 4;$$

This pattern is valid up to some finite δ, and so on. For $Q = 0.7 + \delta$, the payoff is limited by the same value 3 as for $Q = 0.7$ (with the different tail), but for higher quorums, it is not so. However, the pattern is constructed for any Q in the same way. Let us consider, as another example, the scheme for $Q = 0.9$.

First, everything is offered to the junior, up to $N = 10$. Starting with 10 people, the captain can offer the junior nothing and one coin to other members of the crew (called aristocrats), so the payout is $1 \ldots 10$.

Then number 2 is offered nothing, and up to $N = 89$ (or $i = 79$), we have

$$C\,\underbrace{012\ldots(i-1)}_{i}\,\underbrace{(i+1)\ldots(i+1)}_{8}\,i.$$

Starting with $N = 90$, 9 people can be offered nothing: number 2 and eight aristocrats:

$$C\,\underbrace{012\ldots[79]}_{80}\,\underbrace{0\ldots0}_{8}[80]$$

$$C\,\underbrace{012\ldots[73]}_{74}\,\underbrace{0\ldots0}_{7}\,\underbrace{1\ldots1}_{8}\,0$$

$$C\,\underbrace{012\ldots[66]}_{67}\,\underbrace{0\ldots0}_{8}\,\underbrace{1\ldots1}_{7}\,\underbrace{2\ldots2}_{8}\,1$$

$$\ldots$$

$$C\,\underbrace{012\ldots[17]}_{18}\,\underbrace{0\ldots0}_{8}\ldots\underbrace{7\ldots7}_{8}\,\underbrace{8\ldots8}_{7}\,\underbrace{9\ldots9}_{8}\,8$$

$$C\,\underbrace{012\ldots[10]}_{11}\,\underbrace{0\ldots0}_{8}\ldots\underbrace{8\ldots8}_{8}\,\underbrace{9\ldots9}_{7}\,\underbrace{[10]\ldots[10]}_{8}\,9$$

Here and below we use brackets to isolate two-digit numbers, so [10] is one payment of ten coins, not two payments of 0 and 1 coin.

Next, one more can be offered nothing:

$$C\,\underbrace{012\dots[10]}_{11}\,\underline{0}\,\underbrace{1\dots1}_{8}\dots\underbrace{9\dots9}_{8}\,\underbrace{[10]\dots[10]}_{7}\,\underbrace{0\dots0}_{8}[10]$$

$$C\,\underbrace{012\dots[10]}_{11}\,\underline{0}1\,\underbrace{2\dots2}_{8}\dots\underbrace{[10]\dots[10]}_{8}\,\underbrace{0\dots0}_{7}\,\underbrace{1\dots1}_{8}\,0$$

$$C\,\underbrace{012\dots[10]}_{11}\,\overbrace{\underline{0}123}^{\text{new}}\underbrace{3\dots3}_{7}\dots\underbrace{[10]\dots[10]}_{8}\,\underbrace{0\dots0}_{8}\,\underbrace{1\dots1}_{7}\,\underbrace{2\dots2}_{8}\,1$$

$$\dots$$

$$C\,\underbrace{012\dots[10]}_{11}\,\underbrace{\overbrace{012\dots[10]}^{\text{new}}}_{11}\,\underbrace{[10]\dots[10]}_{7}\,\underbrace{0\dots0}_{8}\dots\underbrace{7\dots7}_{8}\,\underbrace{8\dots8}_{7}\,\underbrace{9\dots9}_{8}[10]$$

$$C\,\underbrace{012\dots[10]}_{11}\,\underbrace{012\dots[10]}_{11}\,\underbrace{\overbrace{\;0\;}^{\text{new}}\dots0}_{8}\,\underbrace{1\dots1}_{8}\dots\underbrace{8\dots8}_{8}\,\underbrace{9\dots9}_{7}\,\underbrace{[10]\dots[10]}_{8}\,0$$

The group of eight people who expect 10 coins is offered nothing, as well as the leader (underlined in the scheme) of the group with growing demands (he expects 11) and number 2 until another group (marked in the scheme by the word “new”) of $12\dots[10]$ grows up.

By then it will be possible to exclude one more person, which would be the leader of the new group. The pattern depends on divisibility by 11, with a repeating group of growing demands from 0 to 10 and a tail that depends on the remainder of the division by 11. The maximum payout can be kept no higher than 10 coins, provided that the number of the crew is sufficiently high, although for a small crew it is much higher.

The maximal payoff is asymptotically $(1-Q)^{-1}$. The number of people who can be offered nothing is $(1-Q)N$, and to $N-1$ crew members (with the captain excluded) this can be applied $(1-Q)^{-1}$ times (to those, who desire more than others). So, after getting nothing, the desire of a crew member will grow up to this value.

It is convenient to imagine the growth of the crew by adding new captains. Then the ex-captain becomes number 2 and wishes to become the captain again, so he is not happy with any offer and thus is offered nothing. He joins the group of $\lfloor(1-Q)^{-1}\rfloor+1$ crew members with increasing demands: $012\dots$. The leader of this group, who wants too much, is offered nothing and joins the next such group. So, the length of such groups remains constant. The leader of the right-most group, who are offered nothing, joins the new growing group; after two steps, this ex-leader gets one coin and the new ex-leader joins him with 0 to form the 01 sequence, then one more comes and they form 012 sequence, and so on. The rest of the crew (members of low rank, who are also the oldest) form some pattern, including sequences of members who want the same amount. Among them are the aristocrats and the junior. The captain offers nothing to the group leaders, who want too much, and either to some constant-size groups in the low-rank part, including aristocrats, who want too much each, or to some shorter group plus the junior. Note that by the time the new group is ready, its leader can join the set of those who can be offered nothing.

4 Applications

Similar problems appear in volunteer computing [1]. Assume that owners of computers join to solve a computationally intense problem; the effective power of their devices is different, not only because of different hardware but also because of different fraction of power granted to the project. The owner of the most powerful device, i.e., the one who contributes the most, distributes some sort of credits for the work. Of course, if a significant part of the participants is not happy with this distribution, they may expel the leader (or launch another project with fewer members). The quorum may be different depending on the agreement between the members and psychological description of their behaviour (if many colleagues approve the offer you do not like, perhaps you would obey).

One could assume that the leader would distribute the credits "fairly", but this is not the case: the payoff pattern is highly uneven, and individual payoff depends very weakly on the personal contribution.

5 Conclusion

In this paper, we present the comprehensive quorum analysis in the "Pirate game" problem. We discover six intervals of quorum value with the different strategies patterns; they are:

- $0 < Q \leq 1/3$: the payoff depends on divisibility on 2 and is given by (3)–(4);
- $1/3 < Q \leq 0.5$: the payoff depends on divisibility on 2 and is given by (1)–(2);
- $0.5 < Q \leq 0.6$: the payoff depends on divisibility on 3 and is given by (5)–(7);
- $0.6 < Q \leq 0.625$: the payoff depends on divisibility on 3 and is given by (8)–(10);
- $0.625 < Q \leq 0.66(6)$: the payoff depends on divisibility on 4 and is given by (11)–(14);
- $0.66(6) < Q < 1$: this interval is divided into subintervals in which the patterns are constructed as it is described in subsection 3.5.

The main conclusions of the study are: the captain with enough money to share is always able to make an offer to satisfy enough crew members and to keep most coins for himself; the maximal individual payoff does not depend on the crew size if this size is sufficiently large; and in a general case, the captain has some freedom in choosing those who are offered something and those who are not.

References

1. Anderson, D.P., Fedak, G.: The computational and storage potential of volunteer computing. In: Turner, SJ and Lee, BS and Cai, W (ed.) Sixth IEEE International symposium on cluster computing and the GRID: spanning the world and beyond. pp. 73+. IEEE Comp Soc TCSC; Oracle; HP; IBM; Intel; Sun; Platform; PTC (2006), 6th IEEE International Symposium on Cluster Computing and the Grid (CCGRID 2006), Singapore, May 16–19, 2006
2. Brams, S., Taylor, A.: An envy-free cake division protocol. The American Mathematical Monthly **102**(1), 9–18 (1995)
3. Chi Changye: A game-theoretic analysis of the pirate game. Bachelor Thesis, National University of Singapore (2019)
4. Coram, B.: Second best theories and the implications for institutional design. In: Goodin, R. (ed.) The theory of Institutional Design, pp. 90–102. Cambridge University Press (1996)
5. Fontes, D.F.P.: Quantum Pirates. A Quantum Game-Theory Approach to The Pirate Game. Master's thesis, Tecnico Lisboa (2014)
6. Moulin, H.: Game theory for social sciences. University Press, New York (1986)
7. Stewart, I.: A puzzle for pirates. Scientific American **5**, 98–99 (1999)