Application of the Gröbner Basis Method for the Study of Nonlinear Control Systems

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Abstract

The paper considers methods for estimating stability using Lyapunov functions, which are used for nonlinear polynomial control systems. The apparatus of the Gröbner basis method is used to assess the stability of a dynamical system. To apply the method, the canonical relations of the nonlinear system are approximated by polynomials of the components of the state and control vectors. The equilibrium states of a nonlinear polynomial system are determined as solutions of a nonlinear system of polynomial equations. An example of determining the equilibrium states of a nonlinear polynomial system using the Gröbner basis method is given. The application of the Gröbner basis method for estimating the attraction domain of a nonlinear dynamic system with respect to the equilibrium point is considered. The coordination of input-output signals of the system based on the construction of Gröbner bases is considered.

Introduction

The most of the dynamical systems in technology and nature are nonlinear dynamical systems. The canonical relations of a nonlinear system can be approximated by polynomials of the components of the state and control vectors. Stability testing using the method of Lyapunov functions is widely applied to nonlinear systems.

There are several methods in the literature to identify candidates for Lyapunov functions [Krasovsky59]:

decomposition of the sum of squares [Papach02];

using the Gröbner basis to select parameters [Forsman91];

use of homotopy operators for decomposition of the vector field of states of the system [Chukanov12, Edelen85];

the assumption that the derivative of the Lyapunov function is negative definite, and then obtain by integration and check the positive definiteness (gradient method).

Gröbner bases are used to solve problems in the theory of nonlinear systems. Some of the applications of the Gröbner basis can be named: estimation of equilibrium states of a nonlinear system; finding the critical points of a given nonlinear system with the Lyapunov function; coordination of input-output signals of the system.

Gröbner bases facilitate the solution of a system of multidimensional polynomial equations in the same way as the Gaussian elimination algorithm makes it possible to solve a system of linear algebraic equations. In lexical

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ordering the Gröbner basis has a triangular structure, reminiscent of the triangular structure in the Gaussian elimination method.

The theory of control of dynamic objects can be divided into two subgroups [Khalil02]:

(1) systems in which the principle of superposition operates, and linear control methods can be used;

(2) systems in which the superposition principle does not work, and it is necessary to use nonlinear control methods. To improve the quality of the dynamic object control system, it is necessary to take into account the nonlinear features of the system.

1 Gröbner bases

The objects in the theory of Gröbner bases are polynomial ideals and algebraic varieties [Nesic02]. Let $p_1, ..., p_s$ be multidimensional polynomials in variables $x_1, ..., x_n$, whose coefficients lie in the field k (we will consider the field of real numbers $\mathbb{R}$). The variables $x_1, ..., x_n$ are considered "place markers" in the polynomials: $p_1, \ldots, p_s \in \mathbb{R}[x_1, \ldots, x_n]$. Algebraic variety defined by the polynomials $p_1, \ldots, p_s$ is the collection of all solutions in $\mathbb{R}^n$ of the system of equations:

$$\begin{aligned} &p_1(x_1, ..., x_n) = 0, \\ &\ldots \\ &p_s(x_1, ..., x_n) = 0. \end{aligned} \tag{1}$$

Formally:

$$V(p_1, ..., p_s) := \{(a_1, ..., a_n) \in \mathbb{R}^n : p_i(x_1, ..., x_n) = 0, i = 1, ..., s\} \tag{2}$$

The polynomial ideal I, which is generated by $p_1, ..., p_s$, is a set of polynomials obtained by combining these polynomials by multiplying and adding with other polynomials:

$$I = \langle p_1, ..., p_s \rangle := \left\{ f = \sum_{i=1}^{s} g_i p_i : g_i \in \mathbb{R}[x_1, ..., x_n] \right\} \tag{3}$$

The polynomials $p_i, i = 1, \ldots, s$ form the basis of the ideal I. A useful interpretation of the polynomial ideal I is in terms of the equations (3). Multiplying g_i by arbitrary polynomials $g_i \in \mathbb{R}[x_1, ..., x_n]$ and adding them, we get the consequence from (1):

$$f = g_1 p_1 + \ldots + g_s p_s = 0,$$

and $f \in I$. Therefore, $I = \langle p_1, \ldots, p_s \rangle$ the ideal contains all "polynomial consequences" of the equations (3).

The Gröbner basis method is based on the concept of monomial ordering (a monomial is a polynomial consisting of one term), since it introduces a corresponding extension of the concept of a leading term and a leading coefficient, familiar for one-dimensional polynomials, to multidimensional polynomials. Let's consider lexicographic or lex order [Nesic02]. Let α, β be two n - tuples of integers $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. n -tuple α follows β (in lex order), which is denoted as $\alpha \succ \beta$, if in the difference of vectors $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ the leftmost nonzero element is positive. For the polynomial $f = x_1^3 x_2 x_3^3 + 2x_1^3 x_3^4$ using lex order $x_1 \succ x_2 \succ x_3$ results in $x_1^3 x_2 x_3^3$ follows $x_1^3 x_3^4$, since the multidegrees of monomials satisfy: $(3, 1, 3) \succ (3, 0, 4)$. In this order, the leading coefficient and the leading term are respectively $LC(f) = 1$ and $LT(f) = x_1^3 x_2 x_3^3$. When using lex of order $x_3 \succ x_2 \succ x_1$ senior term: $LT(f) = 2x_1^3 x_3^4$, since $(4, 0, 3) \succ (3, 1, 3)$.

The ideal I has no unique basis, but for any two different bases $\langle p_1, ..., p_s \rangle$ and $\langle g_1, ..., g_m \rangle$ of the ideal I, the varieties $V(p_1, \ldots, p_s)$ and $V(g_1, \ldots, g_m)$ are equal; the variety depends only on the ideal generated by its defining equations. If all polynomials in a given basis of an ideal have a degree lower than the degree of any other polynomial in an ideal, then this basis is the simplest. For an ideal I and a given monomial order, we denote the set of leading terms of elements I as $LT(I)$. The ideal generated by elements from $LT(I)$ is denoted by $\langle LT(I) \rangle$. The Gröbner basis is formally defined as a set of polynomials $g_1, \ldots, g_m$, for which $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_m) \rangle$. When calculating Gröbner bases, a monomial order is specified. We note two properties of Gröbner bases for a given monomial order:

1. Each ideal $I \subset \mathbb{R}[x_1, ..., x_n]$, different from the trivial $\langle 0 \rangle$, has a Gröbner basis.

2. For the ideal $I \subset \mathbb{R}[x_1, ..., x_n]$, different from the trivial $\langle 0 \rangle$, the Gröbner basis of the ideal I can be calculated using a finite number of algebraic operations.

For a given set of polynomials P, there is an algorithm that computes the Gröbner basis for the (ideal generated by) P in a finite number of steps [Buchberger70]. Buchberger's algorithm generalizes algorithms: Gaussian elimination for a system of linear algebraic equations and Euclid's algorithm for calculating the greatest common divisor of a set of one-dimensional polynomials. This algorithm was implemented on computers in symbolic computation programs using Gröbner bases for solving systems of polynomial equations [**?**, Wolfram03, Demenkov15].

2 Finding equilibrium states of a nonlinear dynamical system

The use of the Gröbner basis in finding solutions to a nonlinear system of polynomial equations is similar to the application of the Gauss method for solving a quadratic system of linear equations. Consider an example of reducing a nonlinear system of polynomial equations: $p_1 = x_1 - x_2^2 = 0$, $p_2 = x_2 + x_3^2 = 0$, $p_3 = x_3 - 2x_1^2 = 0$, to a triangular form using the Gröbner basis method for lex order: $x_1 \succ x_2 \succ x_3$. In the WOLFRAM MATHEMATICA package, the function call
$GroebnerBasis[\{p1, p2, p3\}, \{x1, x2, x3\}, \{\}]$
leads to a triangular Gaussian form of polynomial equations:

$$\begin{aligned} &x_1 - x_3^4 = 0, \\ &x_2 + x_3^2 = 0, \\ &-x_3 + 2x_3^8 = 0, \end{aligned}$$

which allows us to get a solution to this system.

Consider a nonlinear system without inputs $\dot{x}(t) = f(x(t)); x, f \in \mathbb{R}^n, t \in \mathbb{R}$, where $f(x) = 0$ is a vector of polynomials in x. The equilibrium states for this polynomial system are obtained as solutions of a nonlinear system of polynomial equations: $f(x) = 0$.

Example 1

Equilibrium states of the [Nesic02] polynomial system:

$$\begin{aligned} &\dot{x}_1 = x_1 + x_2 - x_3^2, \\ &\dot{x}_2 = x_1^2 + x_2 - x_3, \\ &\dot{x}_3 = -x_1 + x_2^2 + x_3, \end{aligned}$$

can be obtained as solutions of system polynomial equations:

$$\begin{aligned} &p_1 := x_1 + x_2 - x_3^2 = 0, \\ &p_2 := x_1^2 + x_2 - x_3 = 0, \\ &p_3 := -x_1 + x_2^2 + x_3 = 0. \end{aligned}$$

The Gröbner basis for the ideal (p_1, p_2, p_3) using lex order: $x_1 \succ x_2 \succ x_3$, has the form:

$$\begin{aligned} &g_1 := 4x_1 - 2x_1^2 - 4x_1^3 + x_1^4 + x_1^6, \\ &g_2 := -x_1^2 + x_1^4 - 2x_2 + 2x_1^2 x_2, \\ &g_3 := -x_1 + x_1^2 + x_2 + x_2^2, \\ &g_4 := -x_1^2 - x_2 + x_3. \end{aligned}$$

Algebraic equations $g_i = 0, i = 1, 2, 3, 4$ has the same solutions as $p_j = 0, j = 1, 2, 3$. The polynomial g_4 depends only on x_3; from the algebraic equation $g_4(x_3) = 0$, you can determine x_3. If the numerical value of x_3 substitute in $g_3(x_2, x_3) = 0$, then you can define x_2; from $g_2(x_1, x_2, x_3) = 0$ you can define x_1.

In the WOLFRAM MATHEMATICA package: Form an ideal of polynomials:
$p1 = x1 + x2 - x3^2$; $p2 = x1^2 + x2 - x3$; $p3 = -x1 + x2^2 + x3$.

Let us define the Gröbner basis:
$grbas = GroebnerBasis[\{p1, p2, p3\}, \{x3, x2, x1\}, \{\}]$,
$grbas = \{4x1 - 2x1^2 - 4x1^3 + x1^4 + x1^6, -x1^2 + x1^4 - 2x2 + 2x1^2x2,$
$-x1 + x1^2 + x2 + x2^2, -x1^2 - x2 + x3\}$.

To find the roots of x_1, we define the reduced Gröbner basis:
$grbas = GroebnerBasis\left[\{p1, p2, p3\}, \{x3, x2, x1\}, \{x3, x2\}\right]$,

$grbas = \left\{4x1 - 2x1^2 - 4x1^3 + x1^4 + x1^6\right\}.$

Let's execute: $Roots[4x1 - 2x1^2 - 4x1^3 + x1^4 + x1^6 == 0, x1]$.

To find the roots of x_2 with known x_1, we define the reduced Gröbner basis:
$grbas = GroebnerBasis\left[\{p1, p2, p3\}, \{x3, x2, x1\}, \{x3\}\right],$
$grbas = \left\{-x1 + x1^2 + x2 + x2^2\right\}.$

Let's execute: $Roots[-x1 + x1^2 + x2 + x2^2 == 0, x2]$.

To find the roots of x_3 with known x_1, x_2, execute:
$Roots[-x1^2 - x2 + x3 == 0, x3]$. The results are shown in Table 1.

Table 1.

	x_1	x_2	x_3
Solution 1 :	$x_1 = -1,$	$x_2 = 0.5 - 1.32i,$	$x_3 = 0.5 - 1.32i,$
Solution 2 :	$x_1 = 0,$	$x_2 = 0,$	$x_3 = 0,$
Solution 3 :	$x_1 = 1,$	$x_2 = 0,$	$x_3 = 1,$
Solution 4 :	$x_1 = 1.18,$	$x_2 = -0.69,$	$x_3 = 0.70,$
Solution 5 :	$x_1 = -0.59 - 1.74i,$	$x_2 = -2.35 + 1.03i,$	$x_3 = -5.04 + 3.09i,$
Solution 6 :	$x_1 = -0.59 + 1.74i,$	$x_2 = 1.35 - 1.03i,$	$x_3 = -1.35 - 3.09i,$
Solution 7 :	$x_1 = -1,$	$x_3 = -0.5 + 1.32i,$	$x_3 = 0.5 + 1.32i,$
Solution 8 :	$x_1 = 0,$	$x_2 = -1,$	$x_3 = -1,$
Solution 9 :	$x_1 = 1,$	$x_2 = -1,$	$x_3 = 0,$
Solution 10 :	$x_1 = 1.18,$	$x_2 = -0.31,$	$x_3 = 1.08,$
Solution 11 :	$x_1 = -0.59 - 1.74i,$	$x_2 = 1.35 - 1.03i,$	$x_3 = -1.35 + 1.03i,$
Solution 12 :	$x_1 = -0.59 + 1.74i,$	$x_2 = -2.35 + 1.03i,$	$x_3 = -5.04 - 1.03i.$

□

3 Application of the Gröbner basis method in the theory of the method of Lyapunov functions

3.1 Estimation of the area of attraction

The set of all initial conditions of a dynamical system, which converge to the same equilibrium state, is called the area of attraction of this equilibrium state [Forsman91, Sidorov19]. One way to get an estimate of the domain of attraction is to use the Lyapunov functions.

The standard result of Lyapunov's theory is that if $x = 0$ is an equilibrium point for a system with continuous time: $\dot{x} = f(x), x \in D \subset \mathbb{R}^n$, is a domain containing $x = 0$ and $V : D \to \mathbb{R}$ is a continuously differentiable Lyapunov function such that $V(0) = 0$ and $V(x) > 0, \dot{V} = V_x f(x) < 0, \forall x \in D - \{0\}$; then the point $x = 0$ is asymptotically stable. For such a Lyapunov function, consider the sets $\Omega = \{x \in \mathbb{R}^n : V_x f(x) < 0\}$ and $B_d = \{x \in \mathbb{R}^n : V(x) \leq d\}$. If there is a value $d > 0$ such that $B_d \subset \Omega$, then the set B_d is an estimate of the domain of attraction.

For polynomial systems with a polynomial Lyapunov function V, the Gröbner basis can be used to determine B_d. You can determine the largest B_d by finding a d such that $B_d \subset \Omega$. For polynomial systems with polynomial Lyapunov functions, $V(x) - d$ and $V_x f(x)$ are polynomials and the boundaries of the sets B_d and Ω are varieties $Z(V - d)$ and $Z(V_x f(x))$, respectively. At the points of contact $Z(V - d)$ and $Z(V_x f(x))$, the gradients V and $V_x f(x)$ are parallel [Luenberger16]. Using this information, we obtain a system of $n+2$ polynomial equations in $n+2$ variables $(x_1, \ldots, x_n, d, \lambda)$, where λ is the Lagrange multiplier (see Appendix):

$$\begin{aligned} &V - d = 0, \\ &V_x f = 0, \\ &\nabla(V_x f) - \lambda \nabla V = 0. \end{aligned} \tag{4}$$

In the case of the vector of Lagrange multipliers $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m$ we obtain a system of $n + m + 1$ equations from $n + m + 1$ variables $x_1, ..., x_n, d, \lambda_1, \ldots, \lambda_m$.

Calculating the Gröbner basis for this system, where the variable d has the lowest rank in the lex order, we obtain a polynomial equation for d. The smallest positive solution of this equation (the value $d_{\min} > 0$), is the best estimate of the area of attraction.

Example 2

Consider a second-order system:

$$\dot{x} = f(x) = \begin{pmatrix} -x_1 \\ -x_2 + 2x_1x_2^2 \end{pmatrix}$$

and choose the Lyapunov function $V(x) = x^T \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} x = 4x_1^2 + 4x_1x_2 + 3x_2^2$, therefore: $V_x = \begin{pmatrix} 8x_1 + 4x_2 \\ 4x_1 + 6x_2 \end{pmatrix}$; $\dot{V} = V_x f = -8x_1^2 + 12x_1x_2^3 - 8x_1x_2 + 8x_1^2x_2^2 - 6x_2^2$.

The fact that the gradients are parallel $(\nabla(V_x f) - \lambda \cdot \nabla V = 0)$ gives additional equations:

$$\begin{cases} g_1 = 8x_1 + 4x_2 - \lambda(-16x_1 + 12x_2^3 - 8x_2 + 16x_1x_2^2), \\ g_2 = 4x_1 + 6x_2 - \lambda(36x_1x_2^2 - 8x_1 + 16x_1^2x_2 - 12x_2). \end{cases}$$

Let us calculate the Gröbner basis for four polynomials $\{V - d, V_x f, g_1, g_2\}$ in ordering: $d \prec x_1 \prec \lambda < x_2$.

This reduces the system to a polynomial: $4d^4 - 147d^3 + 768d^2 + 2048d$, which results in the values of the roots: $\{\ 0 \quad 29.71 \quad -1.92 \quad 8.97\ \}$. The smallest nonzero positive value of d for which there is a solution to the system is $d \approx 8.97$.

In the WOLFRAM MATHEMATICA package: $Vd = 4x1^2 + 4x1x2 + 3x2^2 - d$;
$Vxf = -8x1^2 - 8x1x2 - 6x2^2 + 8x1^2x2^2 + 12x1x2^3$;
$g1 = 8x1 + 4x2 - lam(-16x1 - 8x2 + 16x1x2^2 + 12x2^3)$;
$g2 = 4x1 + 6x2 - lam(-8x1 - 12x2 + 16x1^2x2 + 36x1x2^2)$.
$grbas = GroebnerBasis[\{Vd, Vxf, g1, g2\}, \{d, x1, lam, x2\}, \{x1, lam, x2\}]$,
$grbas = \{2048d + 768d^2 - 147d^3 + 4d^4\}$,
$Roots[2048d + 768d^2 - 147d^3 + 4d^4 == 0, d] \Rightarrow d = \{\ 0 \quad 29.71 \quad -1.92 \quad 8.97\ \}$. $\square$

3.2 Decomposition of the dynamical system vector field $\dot{x} = f(x)$

For a dynamic system: $\dot{x} = f(x); x \in \mathbb{R}^n, f(x) \in \mathbb{R}^n, f(0) = 0$, form a vector field $X = f(x)\dfrac{\partial}{\partial x}$. Form the corresponding differential form $\omega = f(x)dx$ in the dual basis $\left\langle dx_i, \dfrac{\partial}{\partial x_j} \right\rangle = \delta_{ij}$. Let us construct a scalar potential from the vector field X by using the homotopy operator centered at the point $x_0 = 0$ for the form $\omega = f(x)dx$:

$$\mathbb{H}(\omega) = \int_0^1 \left(x\frac{\partial}{\partial x} \right) \rfloor \left(f(\lambda x)dx \right) d\lambda = \int_0^1 x^T f(\lambda x) d\lambda.$$

We will assume that $\varphi(x) = \mathbb{H}\omega(x)$ is a scalar potential.

Example 3

Consider an example of dynamic equations:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2, \\ \dot{x}_2 &= -x_2 - x_1^2. \end{aligned}$$

Let's construct a dual differential form:

$$\omega = (-x_1 + x_2^2)dx_1 + (-x_2 - x_1^2)dx_2,$$

to which we apply the homotopy operator with $x_0 = 0$:

$$\begin{aligned} \varphi(x) = \mathbb{H}(\omega(x)) &= \int_0^1 x^T f\left(\tilde{\lambda}x\right) d\tilde{\lambda} = \int_0^1 \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -\tilde{\lambda}x_1 + \tilde{\lambda}^2 x_2^2 \\ -\tilde{\lambda}x_2 - \tilde{\lambda}^2 x_1^2 \end{pmatrix} d\tilde{\lambda} = \\ &= -\frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{3}(x_1x_2^2 - x_2x_1^2). \end{aligned}$$

Let us choose the function as the scalar Lyapunov function:

$$
\begin{aligned}
&V(x) = -6\cdot\phi(x) = 3\left(x_1^2+x_2^2\right) + 2\left(x_1x_2^2 - x_2x_1^2\right),\\
&\dot V = V_x f = \begin{pmatrix} 6x_1+2x_2^2-4x_1x_2 & 6x_2-2x_1^2+4x_1x_2 \end{pmatrix}\begin{pmatrix} -x_1+x_2^2\\ -x_2-x_1^2\end{pmatrix} =\\
&\quad = -6x_1^2-6x_2^2-6x_2^2x_1+6x_1^2x_2.
\end{aligned}
$$

Let's find the solution of the system $V\_d = V(x) - d = 0$ in the WOLFRAM MATHEMATICA package:

$$
\begin{aligned}
&V\_d = 3x_1^2+3x_2^2+2x_2^2x_1-2x_1^2x_2-d,\\
&V_xf = -6x_1^2-6x_2^2-6x_2^2x_1+6x_1^2x_2,\\
&g_1 = 6x_1+2x_2^2-4x_1x_2+lam(-12x_1-6x_2^2+12x_1x_2),\\
&g_2 = 6x_2+4x_2x_1-2x_1^2+lam(-12x_2-12x_2x_1+6x_1^2),\\
&grb = GroebnerBasis[\{V\_d, V_xf, g_1, g_2\}, \{d, x_1, lam, x_2\}, \{x_1, lam, x_2\}],\\
&grb = \left\{54d-29d^2+d^3\right\}.
\end{aligned}
$$

The roots of the polynomial $54d-29d^2+d^3 are: d_1=0, d_2=2, d_3=27$.
The smallest nonzero positive value d, for which there is a solution to the system: $d_{\min}=2$.

4 Conversions of input-output signals of a nonlinear system

Consider a differential ring - a ring on which the differentiation operation is defined. It is assumed that differentiation is carried out with respect to the implicit variable t. A differential ideal is an ideal that is closed under differentiation.

A polynomial system in the state space is a system of differential equations:

$$\dot x_1 = f_1(x,u), \ldots, \dot x_n = f_n(x,u), y = h(x,u),$$

where $h, f_i \in \mathbb{R}[x,u], \forall i$.

Thus, every polynomial system in the form of a state space corresponds to a differential ideal in $\mathbb{R}[x,u,y]$:

$$I = [\varphi_1, \ldots, \varphi_n, y-h],$$

where $\varphi_i = \dot x_i - f_i(x,u), i = 1,\ldots,n$.

The problem of transformation from the state space to the input-output form: let I be a differential ideal; find a generator for the differential ideal $I \cap \mathbb{R}[u,y]$.

Example 4

Suppose that it is necessary to find a differential relationship between u and y from the description in the state space of the system:

$$\dot x_1 = -2x_1 + x_2^2; \dot x_2 = -x_1x_2 + u; y = x_2.$$

Differentiating the equations of the system with respect to t and replacing $\dot x_i$ by f_i, we get:
$g_1 = y - x_2; g_2 = \dot y - (u - x_1x_2); g_3 = \ddot y - (\dot u - (x_2^2 - 2x_1)x_2 - x_1(u - x_1x_2))$.

Replace $y \to y_0, \dot y \to y_1, \ddot y \to y_2, u \to u_0, \dot u \to u_1$ in g_i, calculate the Gröbner basis G for: $(y_0 - x_1, y_1 - u_0 + x_1x_2, y_2 - u_1 + (x_2^2 - 2x_1)x_2 + x_1(u_0 - x_1x_2))$ relative to lex order: $u_0 \prec u_1 \prec y_0 \prec y_1 \prec y_2 \prec x_1 \prec x_2$.

Therefore, the input signals $u, \dot u$ and the output signals $y, \dot y, \ddot y$ are related by:

$$(-2u - \dot u + 2\dot y + \ddot y + y\dot y)y^3 + (-3u_0^2 + 3u\dot y - \dot y^2)\dot y + u_0^3.$$

In the WOLFRAM MATHEMATICA package:
$g1 = y0 - x1,$
$g2 = y1 - u0 + x1 * x2,$
$g3 = y2 - u1 + (x2^2 - 2 * x1) * x2 + x1 * (u0 - x1 * x2).$
$grbas = GroebnerBasis[\{g1, g2, g3\}, \{u0, u1, y0, y1, y2, x1, x2\}, \{x1, x2\}],$
$grbas = (-2u_0 - u_1 + 2y_1 + y_2 + y_0y_1)y_0^3 + (-3u_0^2 + 3u_0y_1 - y_1^2)y_1 + u_0^3.$
□

Conclusion

The paper considers methods for estimating stability using Lyapunov functions, applied to nonlinear systems. The canonical relations of a nonlinear system are approximated by polynomials of the components of the state and control vectors. To assess the stability, Gröbner bases are used. A method for finding the critical points of a given nonlinear system is proposed. The coordination of input-output signals of the system based on the construction of Gröbner bases is considered.

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