A Little About Models

Igor V. Konnov
Kazan Federal University
Kazan, Russia 420008
konn-igor@ya.ru

Sapienti sat

Abstract

We discuss several aspects of creation of adequate mathematical models in other sciences. In particular, many difficulties stem from great complexity of the source systems and the presence of a variety of uncertain factors. We illustrate the effect of uncertainty on the known consumer demand model. We conclude that not every uncertainty can be represented by a random variable, and that these concepts are not equivalent. We discuss also the role of different information concepts in mathematical models. We give additional illustrative examples of models of quite complex systems.

Key words: Mathematical models; uncertainty; information concepts; mean field games; forest management model.

1 The fallacy of simple concepts

In many sciences, it is customary to create models. Although there are some sciences where most knowledge is obtained through direct observations (and previously these sciences prevailed), but a fundamentally new level can only be achieved by creating models that clarify the essence of studied processes and phenomena.

Since the author's experience is limited to mathematical models, first of all we will talk about them. The role of these models in other sciences, and in other areas of activity is very significant. For example, according to Leonardo da Vinci, "no human investigation can be said true science, if it cannot be demonstrated mathematically". However, there are also opposite opinions about negative influence of mathematical models in other sciences (see, e.g., [Ali80]). These notes can be regarded as an attempt to clarify some important aspects of this issue.

What, in fact, is a mathematical model of some real or imaginary system? A set of relationships that link different parameters and variables and reflect all significant relationships among elements (and subsystems) of this system, as well as the relationships with the system environment. It is obvious that one always try to create any model as simple as possible since it will be used to solve various behavioral assessment tasks related to the system, to estimate the influence of certain parameters on it, etc. I would also like to emphasize that the solutions to all these problems are actually describe properties of the model, although they are usually transferred to the actual system that the model was built on, and this difference is very important for right understanding of the mathematical modeling process.

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A special feature of mathematical models is that they are based on a given formal structure, i.e., enough clearly defined elements and parts of the system under study and all possible interrelations among them, so that they can be written in the form of mathematical relations.

For example, *number* is one of the most fundamental concepts in Mathematics. The use of numbers (count) or the corresponding scalar variable (or parameter) in essence means that all properties of the enumerated objects are not important for the corresponding mathematical model, except for their quantity. That is, they in fact belong to the same type (class) in this model. Of course, some other objects that are more complex than numbers can be also used. But introducing some common preference relation for these general objects also postulates implicitly that they belong to the same class because they allow comparison based on the entered relation.

Based on this, it is quite clear that a formal structure is being introduced automatically explicitly or implicitly already when creating any mathematical model. Of course, such a model can be investigated independently as a mathematical problem, in particular, to find out questions about the existence of solutions, their type, and their dependence on parameters, calculation capabilities, etc. But there is always a question about the application areas of the model. In other words, one has to define what real systems have a structure that it is adequately reflected by the structure of the constructed model. Examples of models that have applications in completely different areas are fairly well known, but the main question is in methods for checking the adequacy of models.

Indeed, incorrect structuring (formalization) of the original systems when creating a mathematical model can lead to useless multi-cost work and, moreover, to false conclusions. At the same time, the successful application of the model in one area does not guarantee its adequacy in another area, or in the case of changing the original real system. In this connection the situation with the use of models in Physics, where mathematical model construction has been carried out for centuries, is rather favorable. Indeed, in many cases it was not just possible to create suitable models of processes and phenomena, but also specify the conditions (ranges) of their successful applications (see, e.g., [SM05, KP83]). In other sciences, the situation is not so favorable in many respects due to the presence of poorly formalized systems (see [EM79]). It should be added that the model that is not structured correctly cannot be improved by using objects with more general properties or more complicated techniques.

A well-known example of incorrect formalization is Ptolemy's geocentric system. Its application to position determination of stars and planets met constantly detected deviations that forced one to use more and more corrections in formulas. Noteworthy, the initial use of the heliocentric system by Copernicus also revealed deviations from the instant positions, but they were caused by an inaccurate shape adopted for orbits (circles instead of ellipses).

Note that various general system structures and their properties are studied separately in the theory of systems, and different approaches are used for this purpose (see, e.g., [Moi81, VVD83]). However, the issues of compliance of structures of the model and real system under study, as a rule, you have to resolve for yourself.

2 Impact of uncertainty

Thus, it turns out that the main problems when building adequate models arise because of the great complexity of the source systems and the presence of a variety of uncertain factors, which makes it naturally difficult to determine the appropriate structure of models. For example, because of the fundamental impossibility to describe behavior of each separate gas molecule in statistical physics, one was able to build an adequate model only at the macro level, i.e. to describe, say, the behavior of a certain volume of the gas as a whole.

Let's illustrate the effect of uncertainty using the *consumer demand model* in the neoclassical theory of market equilibrium (see, for example, [Eke79]). In this model, the consumer's demand of goods defined as a set of solutions to the consumer utility maximization problem on the budget set where prices of goods are specified as external parameters. The key element of this model is just this utility function (or, if necessary, the consumer preference relation), whose existence is deduced from the consumer's ability to accurately compare values of all goods. Such a complete determinism of tastes is quite corresponds to the real behavior of a buyer who went to a local market (or fair) in the Middle Ages held, say, every week, where the same limited set of home-made products with similar properties was proposed for sale every time and almost the same amount of money was used for purchases there (see [Kon16a]). It is clear that a major change in prices could be only invoked by certain external influence on these conditions. However, the specified "marginal" demand model used in neoclassical theory to describe general market behavior within a sufficiently large time interval, with a large number of participants and a wide variety of products, so that the same product produced at a different time or place is considered as a different product. This approach leads to infinite dimensional models of market equilibrium, which are very difficult even to determine the conditions of existence of solutions (see, for example, [ABB90]). It is obvious that the consumer is not able to accurately evaluate the usefulness of all products under these conditions. Aware of this drawback, but trying to keep this model of consumer behavior, as the main properties of the perfect competition model are based on it, supporters of the neoclassical theory proposed to generalize the concept of the utility function based on the so-called "rational expectations", which allowed one to maintain the basic model structure without changes. Meanwhile, the main difference with the usual utility function is that the new does contain undefined factors, and the level of this uncertainty may be arbitrarily high. Therefore, the assumption of maintaining the "marginal" consumer behavior in case of unreliable data seems absolutely unrealistic. For this reason, a fundamentally different type of the model is required to describe consumer's behavior in general.

A fairly popular approach to research and solving problems with uncertainty is the use of random values, so that the uncertainty is simply identified with the randomness in many works. Recall that any random variable is determined on a set (or space) of elementary events together with its probability distribution (normalized measure) on this set, and may be of a continuous or discrete type. However, setting such a measure itself and associated values does not mean that just a random variable is determined, as well as any vector with non-negative coordinates, the sum of which is equal to one is not automatically a probability distribution. This property requires special conditions that justify the possibility of utilization of a probabilistic (stochastic) model.

Namely, setting the probability distribution requires *statistical stability*, i.e. evidence based on multiple observations under the same conditions. Obviously, the statistical stability is achieved only when observing a sufficiently homogeneous and independent process or phenomenon. On the other hand, the presence of a common measure in the form of the probability distribution for elementary events also indicates that these events are of the same type, as noted above about the implicit properties of applying numbers (scalar variables). Thus, the streamlined utilization of probabilistic models for essentially heterogeneous diverse phenomena is incorrect. In addition to the specified conditions, this requires definition of a formal structure that would be adequate to that of the source real system. From this we can conclude that not every uncertainty can be represented by a random variable, and that these concepts are not equivalent.

A fairly standard technique in game theory describing models of conflict situations, i.e. models with uncertain factors, is the utilization of mixed strategies that are nothing but probability distributions on the set of (ordinary) pure strategies, which are then elementary events. This approach leads to a significant complication of the original model, but relaxes sufficient conditions for existence of equilibrium states (see, for example, [Owe95]). The presence of equilibrium states makes the behavior of the described conflict system quite predictable. The implementation of mixed strategies by players consists in conducting a random experiment in accordance with the corresponding distribution and selecting the pure strategy obtained for further actions. This approach looks quite artificial, moreover, it makes sense only in case of multiple repetition of playing this game. Then the players' utilities will tend to their average values, that is, to the game value in mixed strategies. If the game is played once, or a few times, the usefulness of mixed strategies becomes doubtful. In some cases it is possible to utilize the so-called "physical mixture" of strategies (see [Ven72]). For example, if a pure strategy is to select a crop for sowing a field, there is no need to produce a random experiment to select it. One can then just sow the field in proportions specified by the probability distribution. In the general case, players are more likely to will base their actions on any additional information about the other participants, i.e. the game will transform into some multi-stage procedure. Detailed discussion of utilization of mixed strategies for finding solutions of various games can be found in [Ger71, §14,16], [Ger76, §11].

3 Information flows

The model structure for various socio-economic systems, industrial, transport and communication, and in general for systems related to human activity should include information exchange schemes among elements and blocks (subsystems). This is one of the main differences from the structure of models in many natural sciences, such as Physics. Under *information* hereafter will only be understood as its content, rather than its volume recorded on any media. That is, description of elements and subsystems themselves and their relationships is not sufficient for adequate definition of the required system structure without description of the information exchange scheme.

For example, classical models of perfect and imperfect competition describe two different types of decentralized systems in Economics. Separate actions of economic agents (elements of the system) cannot affect the state of the whole system in the Walrasian type perfect competition models. But common actions of economic agents can change the system state. Therefore, each of them in principle need not use information about actions (prices, assortment, volumes) or interests of some other separate agent. Instead, the agents use information about integral indicators of the entire system (for instance, good prices), which may be available to them, although the mechanism for determining common market prices is not clearly defined in the available models (see, for example, [Kon15e]). On the other hand, separate actions of each economic agent in imperfect competition models can change the state of the entire system and, in particular, affect any other economic agent, so the participants will use information about actions and interests of others when choosing their own actions. As a result, it turns out a fundamentally different game-theoretic model, with a different information exchange scheme.

One can also find a lot of examples in history when states with the same government institutions were managed in completely different manners, that is, with different information exchange schemes among these institutions.

In addition to the above (custom) definition, it is also common to define the information concept based on probabilistic representations, which for the difference will be denoted as *information (p)*. For example, let us suppose that the set of all possible events for an investigated system (object) is finite (n) for simplicity, so as the whole set is the union of these n elementary events. A state S of the system is then defined by using some probability distribution $p = (p_1, \ldots, p_n)$ on this set, that allows one to calculate its "entropy"

$$H(S) = -\sum_{i=1}^{n} p_i \log p_i,$$

which is considered as an uncertainty measure of this state. Therefore, the maximal entropy corresponds to the greatest uncertainty, i.e. to the state where all the elementary events may occur with the same probability, or simply where $p_i = 1/n$, $i = 1, \ldots, n$. The probability of events, as usual, can be conditional or unconditional, and the difference between two states is just determined as the information (p); i.e., decrease of entropy gives the positive value of the information (p) in this transition (see, for example, [Str75]). However, this raises a natural question about the generality of such approach, since it in fact states the possibility of existence of one-dimensional representations for any diverse processes and phenomena. For example, reception of some new knowledge about the system invokes transition to a new state, but this knowledge can be somewhat incorrect, so how does one evaluate the change of the entropy in this case? Is it possible to measure the amount of information in any object at all? It is clear that this measurement is only possible within some suitable model with the specified formal structure. In general, both the objects themselves and events with them contain in fact infinite amounts of information, so such "ubiquitous" measurements are meaningless. For instance, when replacing one our car with another of the same kind we clearly understand that it is not the same car. But within the framework of our model (representation) of a car, it will perform the same necessary functions, and all the differences between them, though infinite, are insignificant to us. Hence, we can simply ignore these differences and consider the new car as the same. That's why the amount of necessary information about it can be considered finite. Thus, the information (p) must be related to some specific structure of the model used. Note that different model structures can in principle use the same set of information about one object.

This concept is also associated with the reflected information (data), i.e. recorded on certain media, which can be denoted as *information (d)*. There are many tasks associated with efficient processing, storage, and transmission of the information (d) on various devices that are not directly related to the model creation. We can only note that the information (d) appears initially on the devices within some specific models, and the processing issues are to some extent related to the models and their structures in which they will be used. The differences between these concepts are clearly indicated, for example, in [YY73, p. 111]: "The concept of information arose directly from the problems of communication theory and was specially selected to meet the objectives of this theory. Since the transmission of a fixed length message over a communication line requires approximately the same time and expenses both in the case of an insignificant or even false message and in the case of a message about the greatest discovery, we must assume from the point of view of the communication theory that the amount of information in both these messages is also the same".

4 Additional examples

For more clarity, we will give additional illustrative examples of models of quite complex systems.

4.1 Mean field games

This model is intended to describe the behavior of a team involving a sufficiently large number of dynamic active elements (players), i.e. each of them has its own goal function and state equation that determines the relationship between the trajectories and control functions, as in the custom differential game. The model is based on the assumption that the players differ only in random terms. Then it is suggested to go to the averaged values and after taking the limit on the number of players one can get an optimal control problem, which will consist in maximizing the corresponding averaged functional on the averaged equation of state. Obviously, this approach is based on the direct transfer of modeling principles from statistical physics. But in this case its utilization for socio-economic applications is emphasized (see, for example, [GLL11]). However, this raises a natural question about the validity of the approach where the behavior of, say, gas molecules in a certain volume and behavior of groups of active elements (individuals) with their own interests and sets of actions are considered as the same ones. The mean field game model is in fact based on the assumption that a group of human individuals can be replaced with its generalized representative, whereas all the basic works in social and economic sciences always emphasized the difference in behavior of an individual and a group of individuals.

In particular, the well-known K. Arrow impossibility theorem states that a collective preference relation that is consistent for all the team members is only possible if it matches one of the individual relations under rather general assumptions (see, for example, [Eke79, AHS06]). Therefore, the whole team behavior will be in general more complicated, and it is not represented by some common preference relationship, i.e. a different model is required instead of an optimization problem with respect to some preference relation. This assertion it can be illustrated with simple examples.

Let us consider the election of a team leader when the team consists of n groups ($n > 2$), each i-th group nominates its own candidate a_i and sends its representative to the election with its particular preference relation $\succ_i$. The simplest option for this preference is to consider its own candidate as the best one, i.e. to choose

$$a_i \succ_i a_j, \ \forall j \neq i.$$

Then it is impossible to choose the leader, except to simply take one of these candidates, i.e., to take one of the individual preference relations as the collective. In this case, the share of its support will be equal to $1/n \to 0$ as $n \to \infty$, i.e. the decision will be absolutely "illegitimate".

It is also well-known that collective preferences can be non-transitive, even if all the individual preferences are transitive. It was first noticed by M. Condorcet in the $XVIII$-th century (see, for example, [AHS06]). We now give a generalized version of this paradox. In the previous example, let the representatives compare all the candidates in pairs as follows:

$$\begin{aligned}
&a_i \succ_1 a_{i+1}, \ i = 1, \dots, n-1; \\
&a_i \succ_k a_{i+1}, \ i = 1, \dots, k-2, k, \dots, n-1, \ \text{and } a_n \succ_k a_1, \\
&\ \text{for } k = 2, \dots, n-1; \\
&a_i \succ_n a_{i+1}, \ i = 1, \dots, n-2, \ \text{and } a_n \succ_n a_1.
\end{aligned}$$

If we create the collective preference based on the majority votes rule, we get the cycle

$$a_i \succ a_{i+1}, \ i = 1, \dots, n-1, \ \text{and } a_n \succ a_1.$$

At the same time, the support share for any pairwise preference above is equal to $(n-1)/n \to 1$ as $n \to \infty$.

By themselves, models with a large number of active elements, including those obtained as limit ones from game models are fairly well-known (see [Hil74, ABB90, Mas83, NS83]). But all these models are of equilibrium type and not reduced in general to optimization problems. Hence, they require somewhat different mathematical tools.

4.2 A long-term model for management of renewable natural resources

Since the management of natural resources, including renewable, is essential for sustainable development of our world, the corresponding management tasks are in the spotlight of many researchers. Note that creation of proper models here requires for many heterogeneous, but interrelated factors to be taken into account, in particular, economic, environmental, social ones as well as their outcomes for a fairly long time period. For this reason, the corresponding models are often very complex from the mathematical point of view, and involve uncertain factors. Let us take the long-term forest management model as an example.

First of all, the forests are used for different purposes. In Economics, forest is a source of timber and fuel wood, in Ecology, it is an environment of air purification and carbon absorption, in Biology, it is the habitat of animals, birds, and plants. Also, the forest may be a place of rest from the point of view of human society. Any attempt to combine all these factors into a single goal (utility) function and to specify all the relationships and restrictions, including the effects of weather, environmental pollution, and invasions of harmful insects, etc., will result in an actually unsolvable task.

It is therefore necessary to make some decomposition of the problem into several quite independent blocks. For example, an arrangement of goals with a decomposition of the planning periods was suggested in [Kon14]. Since the sustainable development is the principal goal, it seems better to remove evaluations of economic factors from long-term models. Note that there are a lot of models in this area, which are formulated mainly as optimal control problems, i.e. as problems of maximization of quality criteria along the movement trajectories described by a state equation (see, for example, [AOK09]).

Let us describe this discrete time model; where the time horizon is we divided into stages (years) $t = 1, 2, \dots, T$. The total forest territory is bounded above by S and is divided into many stands, each stand containing only trees of the same age $i = 1, \dots, L$. For simplicity, we suppose that there is only one kind of trees, since the case of many species only increases dimensionality within the same model. We denote by $v^t = (v_1^t, \dots, v_L^t)$ the forest territory vector at the end of the t-th stage, the initial distribution v^0 is supposed to be known. Next, the natural forest dynamics in the absence of external influences can be described by the difference relation

$$v^{t+1} = A(v^t), \quad t = 0, 1, \dots, T-1;$$

where the operator A in the simplest case is given by a matrix of the order L. If necessary, it may contain undefined factors, but this will not affect the essence of the model. We denote by $u^t = (u_1^t, \dots, u_L^t)$ and $w^t = (w_1^t, \dots, w_L^t)$ the harvest and planting territory (square) vectors (say, in hectares) within the t-th time stage, respectively, while $u_i^t = 0$ for $i = 1, \dots, l-1$, and $w_i^t = 0$ for $i = l_0 + 1, \dots, L$, i.e. l is the minimal harvesting age, l_0 is the maximal planting age. Also, L is considered as the maximal product age. Then the dynamics of the forest will be described by the relations:

$$\begin{aligned} & v^{t+1} = A(v^t - u^t) + w^t, \quad t = 0, 1, \dots, T-1; \\ & \sum_{i=1}^{L} v_i^t \leq S, \ v^t \geq 0, \ u^t \geq 0, \ w^t \geq 0, \quad t = 0, 1, \dots, T. \end{aligned}$$

At the end of the planning period, some desired set of feasible distributions is given, for example,

$$v^T \in V,$$

where V is some set in $\mathbf{R}^L$. Among various additional conditions, we impose only lower bounds (i.e. minimal feasible volumes) Γ_t of carbon sequestration per stage t and insert the constraints

$$\sum_{i=1}^{L} \gamma_i v_i^t \geq \Gamma_t, \quad t = 1, 2, \dots, T,$$

where γ_i denotes the carbon sequestration volume from one hectare of forest of age i per stage, for $i = 1, \dots, L$.

The set of feasible trajectories $\{v^t\}$, $\{u^t\}$, $\{w^t\}$, which satisfy the above restrictions, may be rather large. Therefore, one should choose the quality criterion for the management. The traditional approach is to take the profit function along the trajectory. It suffices to determine the timber price (say, per cubic meter) and timber yield of each forest age, the unit cost of felling a hectare of forest of each product age, as well as the unit cost of planting seedlings of each suitable age per hectare. Then one can calculate the profit value along a given trajectory. The main drawback of this approach is that the values l and L are large enough. For the pine forest they are the following: $l \approx 60$ years, $L \approx 120$ years. Then the planning horizon T should be even longer in order to take into account the full rotation age, hence, $T \approx 200$. Therefore, any price values for such a long period of time will be unrealistic, and it is necessary to take a different criterion. For example, one can take a cost comparison based criterion. Let μ_i denote the yield of timber from one hectare of forest of age i and η_i denote the yield from one hectare of forest of age i after attaining age l (or some other reference age). Then one can take define the goal function

$$\sum_{t=1}^{T} \left(\sum_{i=l}^{L} \mu_i u_i^t - \sum_{i=1}^{l_0} \eta_i w_i^t \right)$$

and choose the feasible trajectory that delivers the maximal value of this criterion. If necessary, one can apply uniform criteria. The resulting optimization problems are solved with known computational methods.

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