Application Trend through Planar 3-minimal & Projective Planar 2-minimal Graphs

Volodymyr Petrenjuk[1] and Dmytro Petrenjuk[2]

[1] *Centralukrainian national technical university, Universitetskyj 8, Kropivnytskyj, 25006, Ukraine*
[2] *V.M. Glushkov Institute of Cybernetics of the NAS of Ukraine, Glushkova 40, Kyiv, Ukraine*

Abstract

The mathematical method φ-transformation in which large structures are regarded as a set of small and simple substructures. For this, they may have some common parts, that can be identified and amalgamated when constructing or reconstructing an entire structure from a finite number of substructures. Task: To demonstrate the possibilities of this method for the constructing of the set of all nonisomorphic 3-minimal plane simple graphs in which the set of all vertices is located on the boundaries of three 2-cells and constructing the set of all nonisomorphic 2-minimal projective planar graphs in which the fixed set of points is located on the two boundaries 2-cell or pseudo cells. Based on the method of φ-transformations of the 3-minimal planar graphs and the 2-minimal projective planar have been established and modified algorithms of producing these graphs have been suggested. The main result: algorithms and lists of the 3-minimal planar graphs and 2-minimal projective planar with genus 0 or 1.

Keywords

Graph representations, geometric and topological aspects of graph theory, projective planar

1. Introduction

Let's solve the problem of analysis of a complex system synthesized from the studied simpler substructures in general and their application in computer sciences. We offer a graph-theoretical approach as a way of machine thinking or operating with artificial images-structures. It is known that there are mathematical methods by which large systems as structures are considered through a set of small and simple substructures, which may have some common parts that can be identified in the construction or reconstruction of the whole structure from a finite number of substructures.

The main tool is the φ-method of creating a graph model obtained in the form of a pair of finite sets: sets of vertices and sets of edges to determine the relationships between the structure of vertices as objects. An example of this is the transformation of the main problems of system programming into problems of graph theory with the mathematical support of algorithms.

The main idea of the method of φ-transformation can be interpreted as a way of inheriting a certain property of substructures in the whole structure depending on the properties of the connection (identification of given parts of pair of substructures).

Task. To demonstrate the possibilities of this method for the constructing of the set of all nonisomorphic 3-minimal planar graphs in which the set of all vertices is located on the boundaries of three 2-cells and constructing the set of all nonisomorphic 2-minimal projective-planar graphs in which the fixed set of points is located on the two boundaries 2-cell or pseudosells.

COLINS-2022: 6th International Conference on Computational Linguistics and Intelligent Systems, May 12–13, 2022, Gliwice, Poland
EMAIL: petrenjukvi@i.ua (V. Petrenjuk); dmytrotheukrainian@ukr.net (D. Petrenjuk)
ORCID: 0000-0001-7313-9642 (V. Petrenjuk); 0000-0002-8566-0256 (D. Petrenjuk)

The solution to our task was based on the method of graph transformations, whose founder is M.P. Khomenko, and the concepts he introduced. The recognition of φ-transformations of graphs was taken from [1].

The structural properties of a complex system model presented in the form of a graph model can be studied using a simple graph G with a fixed set of points embedded in the surface on which the edges of the graph will be located, where S is the Euclidean plane or projective plane. The point is either vertex G, or the inner point of the graph of edges G. Let us consider the connected simple graph G, $G=(G^0,G^1)$, where G^0 is the set of vertices and G^1 is the set of edges without multiple edges and without loops as its 2-cell minimal embedding in the surface S. The property of minimality of the model graph over S will be that the graph G with the edge removed or the edge compressed into a point will have changed the specified numerical measure of the fixed set of points of the graph G. For example, model G such a property outer-planarity of the set of all vertices which located on the boundary of one cell is the presence of subgraphs homeomorphic to K_4 or $K_{2,3}$.

This result will be useful in the systematic analysis of both graph models and their topological aspect which will have common properties at the edges and vertices of the graph model.

The cylindrical graphs were introduced in [2]. There was investigated from the point of view of their external planarity and a complete list of 38 graphs characterized by non-cylindrical graphs as minor ones was obtained.

Results: 1) The algorithm and the list of 3-minimal graphs, namely their characterization by the method of φ-transformation of graphs, was given in [3], the list of 32 3-minimal graphs is given in [4]. 2) The algorithm and the list of 34 2-minimal projective planar graphs with a fixed set of points of this graph's nonorientable genera 0 or 1 are presented here.

Application trend. An example of a possible application is the set of points placement problems or automatic control with subsequent access to its points. If we talk about the surface as an almost infinite set of values of the function of several variables on a given finite subset as a set of vertices whose relationship between pairs of elements as a set of edges, we have an almost embedded graph in the surface. If it is possible to set edges as an almost infinite subset of points and in the absence of intersection of edges in infinite vertices of edges, we will have an almost exact embedding of the graph in the surface. If the surface is spherical or resembles some extent a plane without holes, then use the following list of 3-minimal graphs to place on the boundaries of three cells of all vertices of the graph. If the surface is a sphere with a hole or to some extent resembles a plane of holes, then we use the following list of 2-minimal projective planar graphs to place on the boundaries of two cells of a given set of vertices of the graph.

2. 3-Minimal planar graphs and non-cylindrical graphs.

According to [3] we will consider a simple graph G to be a planar graph having the following properties: 1) three 2-cells at the borders of which all vertices of the graph G are located, 2) removal of any edge or its pinching into the point of this edge leads to the destruction of property 1).

According to [2] we define a non-cylindrical graph NCG as a flat-integrated graph NCG having more than two 2-cells of the cylindrical graph on their boundaries where all the vertices of NCG are located. Whereas removal of any edge of graph NCG or squeezing the point of this edge leads to the distribution of all the graph vertices on the boundaries of two 2-cells of the cylindrical graph.

Problem. Let us study the identity of non-cylindrical and 3-minimal dense graphs and compare the graphs from the given lists in figure 1 with the list [2] and the modified algorithm of building all 3-minimal planar graphs.

History of the problem. In [7] there is a short review of works on this problem and the similar problems of shunting of the lists of graphs which would play a role of non-conserved (with the accuracy to homeomorphism) subgraphs for the input graphs, which are checked for the presence of an analogous "external planarity" property for some surfaces.

We have the following relation for the planar graphs:

Proposition 1. All graphs from the list [4] are in the list [2], non-cylindrical and 3-minimal graphs are equivalent, and graphs θ_6, θ_7, K_5, $K_{3,3}$ from the list [2], are absent in the list [4].

We consider the modification of algorithm [4] for constructing three-minimal planar graphs, which was based on the inexact result of characterization of planar graphs with all significant edges with respect to the number of an auxiliary multiplicity of vertices equal to 3 at the operation of removing the remaining edge. The main idea is that such graphs have at least one homeomorphic graph subgraph and at most three such graphs; it is necessary to define the nature and possible variants of combining them.

2.1. The mathematical base for 3-minimal planar graphs.

Theorem 1. If the connected planar graph G has the following condition $(\forall u)(u \in G^1)(t_{G\setminus u}(G^0) = t_G(G^0) - 1 = 2)$, then one of the following propositions holds:

1) $G = K_4^{'}$. The graph $K_4^{'}$ is a graph K_4 in which each edge is 1-divisible;

2) There is a graph φ-transformation $\sum_{i=1}^{2} G_i$ into a graph G defined as follows $\varphi(\sum_{i=1}^{2} G_i, \sum_{j=1}^{n}(y_{1j} \dot{+} y_{2j})) \to (G, \{y_j\}_{j=1}^{n})$ and which satisfies the following conditions:

a) $(\forall i, i = 1,2)[(G_i \neq K_4^{'}) \wedge ((G_i \cong K_{2,3}) \vee (G_i \approx K_4))]$;

b) $G_i(\{y_{ij}\}_{j=1}^{n}) = C_{G_i}^{n-1}(y_{i1}, y_{in})$ - the simple path of the length $n-1$ of the graph G_i, where $y_{i1} \neq y_{in}$, $\{y_{ij}\}_{j=1}^{n} \subset G_i^0 \cup G_i^1$, where $i = 1,2$, (if $n = 0$ then the simple path is formed into a point y_{i1});

c) $G(\{y_i\}_{j=1}^{n})$ - the simple path of the graph G of length $n-1$ ($n = 0$ the simple path is generated to a point y_1);

3) Existence of φ-transformation of the graph $G_0 + G_3$ in the graph G by the following:

$\varphi(G_0 + G_3, \sum_{j=1}^{n}(Z_{0i} \dot{+} Z_{3j})) \to (G, \{Z_j\}_{j=1}^{n})$, where graph image satisfies the following conditions:

a) The G_0 is an $\varphi-$ image of graph $\sum_{i=1}^{2} G_i$ written as in statement 2) of this theorem;

b) $G_3 \approx K_{2,3}$; $G_0(\{Z_{0j}\}_{j=1}^{n})$ is a cycle of the length n (possibly with diagonals) of the graph (possibly the boundary of the outer boundary of the graph $f(G_0)$) ( ), where $f \mid G_0 : G_0 \to \sigma$ is the contribution that realizes $t_G(G_0^0)$, $\{Z_j^0\}_{j=1}^{n} \subset G_0^0 \cup G_0^1$;

c) $G_3(\{Z_{3j}\}_{j=1}^{n})$ is a simple cycle of graph G_3, possibly with diagonals.

4) There is a φ-transformation of the graph $\sum_{i=1}^{2} G_i$ into a graph G defined as follows: $\varphi(\sum_{i=1}^{2} G_i, \sum_{j=1}^{n}(y_{1j} \dot{+} y_{2j}) + (y^{\bullet}{}_{1j} \dot{+} y^{\bullet}{}_{2j})) \to (G, \{y_j\}_{j=1}^{n} \cup \{y^{\bullet}\})$ and satisfies the following conditions:

a) $(\forall i, i = 1,2)[(G_i \neq K_4^{'}) \wedge ((G_i \cong K_{2,3}) \vee (G_i \approx K_4))]$;

b) $G_i(\{y_{ij}\}_{j=1}^{n}) = C_{G_i}^{n-1}(y_{i1}, y_{in}) + y^{\bullet}{}_{i1}$ - simple path length $n-1$ graph G_i is connected with $y^{\bullet}{}_{i1}$ - the isolated graph G_i point which does not belong to the subgraph $G_i(\{y_{ij}\}_{j=1}^{n}) = C_{G_i}^{n-1}(y_{i1}, y_{in})$ of $y_{i1} \neq y_{in}$, $\{y_{ij}\}_{j=1}^{n} \subset G_i^0 \cup G_i^1$, $i = 1,2$, ($n = 0$ the simple path is generated in the point y_{i1} of , at that $y^{\bullet}{}_{i1} = y_{i1}$);

c) $G(\{y_j\}_{j=1}^{n})$ - simple path of graph G with length $n-1$ which no include in subgraph $G_i(\{y_{ij}\}_{j=1}^{n})$).

Proof. Let simple graph G is the connected planar with all significant edges with respect to the number of reachability of the set of vertices equal t 3, in the operation of removing an arbitrary edge and embedding f, $f : G \to \sigma$ set the attachment that implements $t_G(G^0), t_G(G^0) = t = 3$,

$S_G(G^0)=\{s_i\}_{i=1}^3$ -set of 2-cells on the border of which all vertices of the graph G. We denote by the $M(G)$ set of all the different subgraphs H of the graph G constructed for each pair $(s_i s_j)$, where $i \neq j$, of 2-cells from the set $S_G(G^0)$ as the smallest part of the graph G that satisfies the ratio:

$$[((G^0 \cap ds_i \subset H_{ij}^0) \wedge (H_{ij}^0 \cap (ds_j - ds_i) \neq \varnothing)) \vee [((G^0 \cap ds_j \subset H_{ij}^0) \wedge (H_{ij}^0 \cap (ds_i - ds_j) \neq \varnothing)) \wedge (H_{ij} \cong K_4) \vee (H_{ij} \cong K_{2,3}))] \quad (*).$$

Denote by $M'(G)$ - the least included a subset of the set $M(G)$, consisting of the smallest included subgraphs H_{ij} of the graph G, or parts of these subgraphs that satisfy the following conditions:

a) $G^0 \subseteq \bigcup_{\forall H' \in M(G)} (H')^0$;

b) If the subgraph H_{ij} (or its part) is homeomorphic to the graph which, or all the edges of the graph are 1-subdivided or no edge of the graph K_4 is 1-subdivided. In the future, if no reservations are made, we will assume that, with respect to the elements of the set M' and the term "subgraph" of the graph G, does not preclude the fact that this element may be part of the graph.

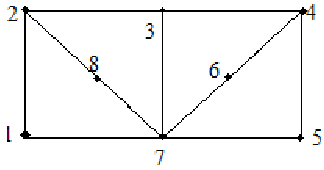


Figure 1. The graph G for constructing the set M.

For example, consider the following embedding of a graph G in an Euclidean plane (fig. 1) and distinguish two sets $S_i=\{s_{ij}\}_{j=1}^3$, $i=1,2$, namely a) $ds_{11}=\{1,2,7,8\}$, $ds_{12}=\{2,3,7,8\}$, $ds_{13}=\{4,5,6,7\}$; б) $ds_{21}=\{1,2,7,8\}$, $ds_{22}=\{1,2,3,4,5,6,7\}$, $ds_{23}=ds_{13}$. For each of them we construct a set M_i, $M_i=M(G)$, $H_{12}^0=\{1,2,3,7,8\}$, $H_{21}=\{1,2,7,8\}$, $H_{23}^0=H_{32}^0$; $(H_{13}^0)'=\{1,2,6,7,8\}$, $(H_{13}^0)''=\{1,2,4,7,8\}$, $(H_{13}^0)'''=\{1,2,5,7,8\}$, $H_{13}^0 \in \{(H_{13}^0)',(H_{13}^0)'',(H_{13}^0)'''\}$, $H_{32}^0=\{3,4,5,6,7\}$, $H_{31}^0=\{1,4,5,6,7\}$, $H_{12}^0=\{1,2,7,8,3\}$, $H_{13}^0=\{1,2,8,7,6,4,3\}$, $H_{21}^0=H_{12}^0$, $H_{23}^0=\{1,2,3,4,5,7,6\}$, $H_{32}^0=\{4,5,6,7,3\}$, $M(G)=\{H_{12},H_{13},H_{23},H_{32}\}$. It is easy to see that the above sets a) and b) exhaust all nonisomorphic sets $S_G(G^0)$, and as a result $M'(G)=\{H_{12},H_{32}\}$.

We prove Proposition 1). Since the graph K_4' is a graph K_4 with all 1-subdivisions edges and will have the property: $(\forall u)(u \in (K_4')^1)[(t_{K_4' \backslash u}((K_4')^0)=2) \wedge (t_{K_4'}((K_4')^0)=3)]$, then we will have the inclusion $K_4' \subset G$. On the other hand, if $K_4' \subset G$, then there is an edge u, $u \in G \setminus K_4'$. Proposition 1) is proved.

Let the graph G be nonisomorphic to the graph K_4'. The following two cases are possible: m), $|M|=m$, where $m=2,3$. Let there be a case 2). Suppose there is equality $M=\{H_i\}_{i=1}^2$. Due to the flatness of the graph G, we have three options φ-transformation of two graphs from the set M defined by one of three options: 1) for two simple chains, 2) for two different pairs of simple chains, 3) for two simple cycles. The first option is called linear in a simple chain, the second nonlinear in two simple chains, the third in a simple cycle. We prove Proposition 2). We have the following two options of the φ- transformation of two subgraphs from the set M: 1) two simple chains, 2) two different pairs of simple chains. That is, we have the following relationship: $G[\bigcap_{i=1}^{2} H_i] = \sum_{j=1}^{n} C_G^{n_j}(a_j, b_j)$ (b1),

where $n \geq 0$, (possibly $n_j = 0$, then a simple chain degenerates into a point a_j). Since, $G \neq K_4^{'}$ the relation b1) implies the existence the φ-transformation of a graph $\sum_{i=1}^{2} H_i$ into a graph G, given as follows: $\varphi(\sum_{i=1}^{2} H_i, \sum_{i=1}^{n}(\sum_{j=1}^{n_1+1} C_{1ij} \dot{+} C_{2ij})) \to (G, \{\{a^*{}_{ij}\}_{j=1}^{ni}\}_{i=1}^{n})$, (A)

where $C_{H_k}^{n_i}(a_{k1}, a_{k(n_i+1)})$ is a simple chain of a subgraph H_k of length n_i with finite vertices $a_{ki1}, a_{ki(n_i+1)}$, where $i = 1(1)n$, $k = 1,2$. Note that the set $\{a_{kij}\}_{j=1}^{n_i+1}$ consists of the vertices of the graph H_k and the inner points of its edges, $\{a^*{}_{ij}\}_{j=1}^{n_i+1}$- the set of vertices of a simple chain $C_G^{n_i}(a^*{}_{i1}, a^*{}_{i(n_i+1)})$ of graphs G with finite set of vertices a^*_{i1}, $a^*{}_{i(n_i+1)}$, such that $\varphi(a_{1ij} \dot{+} a_{2ij}) = a^*{}_{ij}$, $j = 1(1)n_i$, $i = 1(1)n$. For $n = 1$ option 1) and statement 2) in this case is proved. For $n = 2$ we have option 2). We prove the right-hand side of the next double inequality $0 \leq p_1(L(\sum_{i=1}^{2} H_i, G)) \leq 1$, because the left-hand side is trivial. To do this, use the method of proving the opposite.

Assume that for a graph L - the φ-transformation of a graph $\sum_{i=1}^{2} H_i$ into a graph G given in (A), where $L = L(\sum_{i=1}^{2} H_i, G)$ and the inequality holds: $p_1(L) > 1$. Then the graph L will have at least two simple circles. Each of this circles will mean the execution of the 2) φ - transformations at points on at least three different pairs of simple chains without common points. The first elements of each pair will belong to a simple cycle z of graph H_1. The second elements of each pair will belong to a simple cycle $z^{'}$ of the graph H_2. As a result, at least three new 2-cells s with boundaries simple cycles, which cannot all together be the boundaries of two 2-cells s_j, where $S_G(G^0) = \{s_j\}_{j=1}^{3}$, $j = 2,3$. This means that at least one edge of the graph G belonging to z_i one of the loops will not belong to the intersection of two 2-cells s_j, s_i ,, which belong to the set $S_G(G^0)$ will be insignificant relative $t_G(G^0)$ to the operation of its removal. Thus we will have a contradiction to the condition of 3-minimality of the graph G. Since our assumption is incorrect, we have inequality $p_1(L) \leq 1$, which proves the double inequality. Since in Proposition 2) we have a transformation on one pair of simple chains, the proof is complete.

We prove Proposition 3). Let's put $M = \{H_i\}_{i=1}^{3}$. For φ-transformation of a graph $\sum_{i=1}^{3} H_i$ into the graph G, only the following two types are possible:

a) the φ-transformation of type (A), given in the same way as the φ-transformation of a graph $\sum_{i=1}^{3} H_i$ into a graph G, ie on the edges (or parts of edges) of graphs H_i, $i = 1(1)3$, has the property that - the image of the graph $\sum_{i=1}^{2} H_i$ has at least one edge insignificant relative $t_G(G^0)$. And this property will be regardless of whether or not the graph $L(\sum_{i=1}^{3} H_i, G)$ has cycles;

b) φ - transformation of non-type (A) graph $\sum_{i=1}^{3} H_i$ into a graph G, ie it is given so that some φ - images of graphs have common simple cycles. Each pair of φ -images of the graphs H_i, H_j of the set $M^{'}(G)$ can have no more than one common simple cycle. Then the following statements are made with precision to the renumbering of the elements of the set $M^{'}(G)$:

1) There are elements $\varphi(H_i)$, $i = 1,2$, of the set $M^{'}(G)$ with a common cycle and homeomorphic graphs $K_{2,3}$ that do not have common simple loops with the element $\varphi(H_3)$;

2) There are elements $\varphi(H_i)$, $i = 1,2$, that do not have common simple cycles, and an element homeomorphic $K_{2,3}$ has a common simple cycle with an element $\varphi(\bigcup_{i=1}^{2} H_i)$.

Proposition 3) is proved. We prove Proposition 4). The proof will follow as a partial case from the above proof of statement 2) and will differ in that part concerning the necessary condition of degeneracy at the point of simple chains of the second pair.

The proof of theorem 1 is complete.

2.2. Algorithm for constructing 3-minimal planar graphs.

The modification of the algorithm for constructing all 3-minimal plane graphs is based on Theorem 1 and will have the following form:

Input data: The set L_1 of all nonisomorphic chains of graphs for each of the graphs K_4, $K_{2,3}$, ordered by their length and marked for which pair of graphs the chain is taken;

Output: the set of all 3-minimal graphs G;

1. Construct a set L_2 from all different pairs of chains of the set L_1 and a set L_3 from all different two pairs of chains from L_1, as well as a set L_4 composed of different pairs of elements of the set L_1 that generate simple loops without diagonals in columns K_4 or $K_{2,3}$;

2**. While** the set L_1 is not empty to perform the following actions:

2.0. Take the element x from L_1, enter the element x in the list B_1;

2.1. L_1: = L_1 \ x;

2.2. **While** the set L_1 \ B_1 is not empty to perform the following actions:

2.2.1. Take the element u from L_1 \ (B_1 + B_2), enter the element u in the list B_2;

2.2.2. We perform the identification of pairs of vertices or points of pairs of graphs (K_4, K_4), or (K_4, $K_{2,3}$), or ($K_{2,3}$, K_4), or ($K_{2,3}$, $K_{2,3}$), indicated as vertices or points of chains pairs (x, u), for all types of possible φ-transformations of the selected pair of graphs and we obtain a graph G;

2.2.3. *Procedure (G):* Define the reachability number t of the set of all vertices of graph G as the minimum number of simple cycles covering the set of all vertices of graph G.

2.2.4**. If** t = 3 **then** perform:

2.2.4.1. **for** each edge e of the graph G perform in the loop the contraction edge e to a point G: = Ge, perform the procedure 2.2.3;

If t = 3 **then** perform the end of the cycle on the edges of the graph G,

else we derive the graph G;

end of the cycle on the edges e;

2.3. end of the internal cycle;

3. end of the external cycle;

4. **While** the set L_2 is not empty to perform the following actions:

4.0. Take the element x from L_2, enter the element x in the list B_3;

4.1. L_2: = L_2 \ x;

4.2. **While** the set $L_2 \setminus B_2$ is not empty to perform the following actions:

4.2.1. Take the element u from $L_2 \setminus (B_3 + B_4)$, enter the element u in the list B_4;

4.2.2. We perform the identification of pairs of vertices or points of pairs of graphs (K_4, K_4) or (K_4, $K_{2,3}$) or ($K_{2,3}$, K_4) or ($K_{2,3}$, $K_{2,3}$). They are indicated as vertices or points of two different pairs of chains x and u performed on all types of possible φ-transformations for the selected pair of graphs and we obtain the graph G;

4.2.3. *Procedure (G):* Define the reachability number t of the set of all vertices of the graph G as the minimum number of simple cycles covering the set of all vertices of the graph G;

4.2.4. **If** t = 3, **then** perform:

for each edge e of the graph G perform the operation of contraction to a point ;

G: = Ge; perform the procedure 4.2.3;

If t = 3, **then** perform the end of the cycle on the edges of the graph G,

else derive the graph G;

else the end of the cycle on the edges;

4.3. end of the internal cycle;

5. end of the external cycle;

6. **While** the set L_4 is not empty to perform the following actions:

6.0. Take the element z from L_4, enter the element x in the list B_4;

6.1. L_4: = $L_4 \setminus z$;

6.2. **While** the set $L_4 \setminus B_4$ is not empty to perform the following actions:

6.2.1. Take the element u from $L_2 \setminus (B_5 + B_4)$, enter the element u in the list B_5;

6.2.2. We identify pairs of vertices or points of pairs of graphs (K_4, K_4) or (K_4, $K_{2,3}$) or ($K_{2,3}$ K_4) or ($K_{2,3}$, $K_{2,3}$) indicated as vertices of different pairs of cycles, performed on all possible φ-transformations for the selected pair of graphs and get the graph G;

6.2.3. *Procedure* (G): Determine the minimum number t of simple cycles covering the set of all vertices of the graph G;

6.2.4. **If** t = 3 t**hen** perform:

For each edge e of the graph G perform the contraction operation to a point;

G: = Ge, perform the procedure 6.2.3;

If t = 3 **then** perform the end of the cycle on the edges of the graph G,

else derive he graph G;

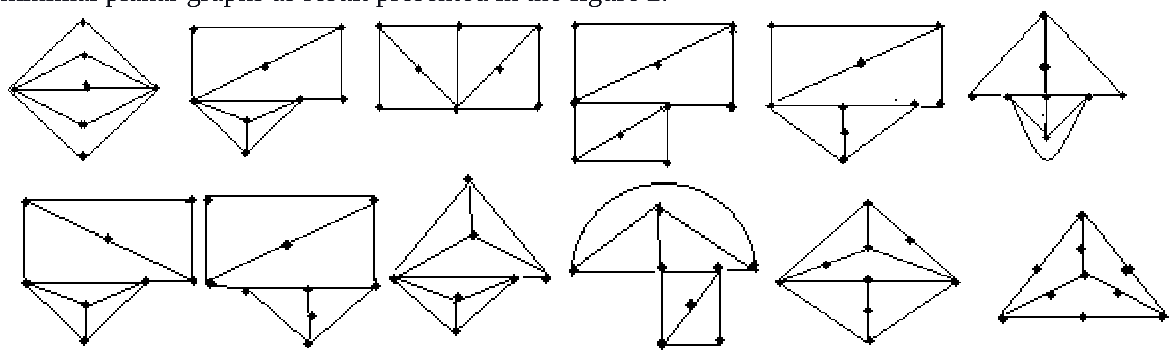
end of the cycle on the edges;

6.3. end of the internal cycle;

7. end of the external cycle;

8. end of the algorithm.

Result: Identity of non-cylindrical graphs to 3-minimal graphs with the proof of equivalence of non-cylindrical and 3-minimal planar graphs, theorem 1 on the characterization of 3-minimal planar graphs, and a modified algorithm for constructing all 3-minimal planar graphs. The 38 diagrams of 3-minimal planar graphs as result presented in the figure 2.



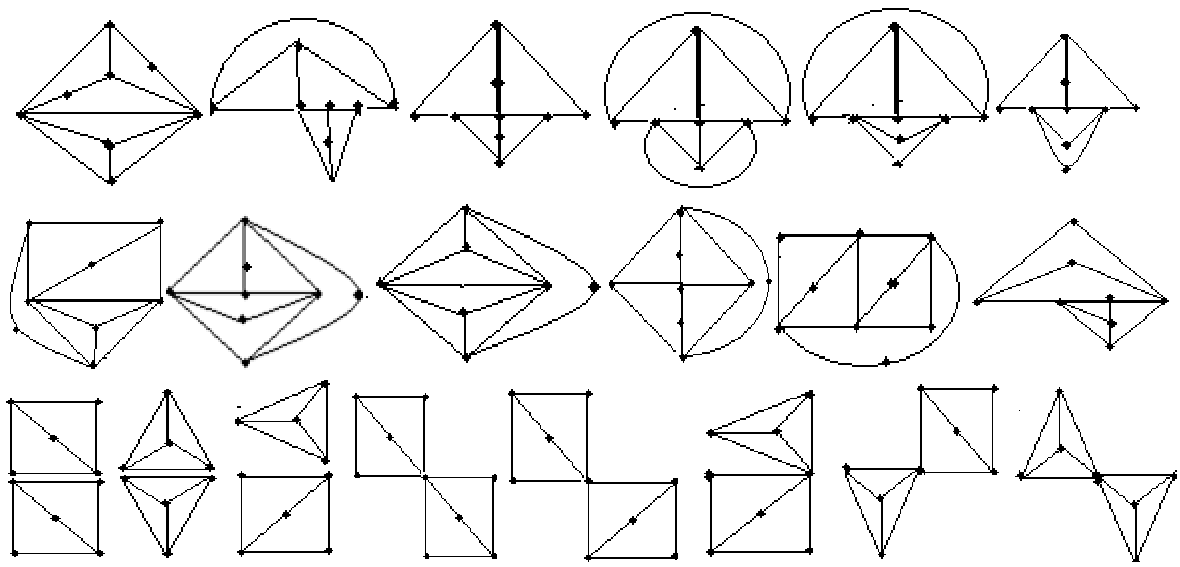


Figure 2. The list of 3-minimal planar graphs.

3. 2-Minimal projective planar graphs.

Task: To construct the set of all nonisomorphic 2-minimal projective-planar graphs in which the fixed set of points is located on the two boundaries of 2-cells or pseudo-cells. Similarly of this task was the tasks for graphs with number of vertexes less then 10 on various genus, which solved in [6], [7], [8]. Introduce a new characteristic that measures the some structure of the set X of points of graph G on S.

Definition 1. For a given embedding f , $f: G \to S$, a graph G in S and a given set of points X , $X \subset G^0 \cup G^1$ determine $t_G(X,S,f), t = t_G(X,S,f)$, the number of reachability of the set X relative to S , if there is a set $S_G(X), S_G(X) = S \setminus f(G)$, which satisfies the condition: $(f(X) \subseteq \bigcup_{i=1}^{t} \partial s_i \cap X) \wedge (f(X) \not\subset \bigcup_{i=1, i \neq j}^{t} \partial s_i \cap X)$, $j = 1,2,...,t$. We say that the set X has a reachability number t, $t_G(X,S) = t$, relative to S , if among all no isomorphic embedding's $f: G \to S$, the number t is the smallest among the numbers $t_G(X,S,f)$. We consider further the set X of points of the graph G t-non-planar concerning the surface S, or (t,S) -non-planar, if $t \geq 2$, where $t_G(X,S) = t$. If $t = 2$, S is a projective plane, and the set X is the set of vertices of the graph G , $X = G^0$, then we will call the graph G non-outer projective planar. A graph G is outer-projective-planar if embeds on the projective plane with all vertices on the boundary of one distinguished cell.

Definition 2. Suppose the embedding f , $f: G \to S$, of the graph G in surface S , which implements t, $t_G(X,S) = t$, where $S_G(X) = S \setminus f(G)$ $S_G(X) = \{s_i\}_1^t$. We will say that concerning a given surface S the set X will have the characteristic $\theta_G(X,S,f)$, $\theta_G(X,S,f) = \theta$, $\theta \geq 1$, if there are θ three cells $\{s_i\}_1^3$ from the set $S_G(X)$, on the boundaries of which the subsets X_i , $X_i \subseteq X$, are placed arbitrarily and satisfy the relation: $G^0 \cap \partial s_1 \cap \partial s_2 \supseteq \{a_1\} \wedge G^0 \cap \partial s_2 \cap \partial s_3 \supseteq \{a_2\} \wedge G^0 \cap \partial s_1 \cap \partial s_3 \supseteq \{a_3\}$, and generates the smallest subgraph G' of the graph G , (possibly degenerate), contains the points $\{a_i\}_1^3$ of pairwise intersection of cell boundaries $\{s_i\}_1^3$. The set X will have the θ-characteristic $\theta_G(X)$ if $\theta_G(X) = \max \theta_G(X,f)$, where the maximum is taken for all embedding's $f: G \to S$, realizing $t_G(X,f) = t$ and $\theta = \theta_G(X,f)$.

Proposition 3.1. Each non-outer projective planar subgraph with a given set of points that are not located on the boundary of one 2-cell or pseudo-cell of an arbitrary graph obstruction of the projective plane can be represented as 1-subdivision graph K_4, or a φ-image of a pair of graphs homeomorphic to graphs from the set {K_5, $K_{3,3}$, K_4, $K_{2,3}$, $K_5 \setminus e$} when identifying pairs of points from the connection sets both in a path and in a cycle.

3.1. The algorithm 1 and its mathematical base.

Theorem 3.1. [5]. The graph G is non-outer projective planar if and only if then $G = H \setminus v$, where v is a vertex of graph-obstruction H of the projective plane N_1.

Theorem 3.2. [9] is the mathematical base for the algorithm 1 for the construction of all no isomorphic non-outer projective plane graphs. The list of minimal projective planar graphs with genus 0 or 1 and fixed subsets of vertices witch has reachability number 2 or 3 is the results of the following modified polynomial algorithm 1.

Begin of Algorithm 1.

Input: The set P of 35 minors P_i of the projective plane N_1 with equivalence classes l_{ij} where $P_i^0 = \sum_{j=1}^{n_i} l_{ij}$, $n_i \leq |P_i^0|$.

Output: List X of graphs.

1. $X := \varnothing$; $v := 0$;
2. **For** i =1 step 1 to 35, **does** these steps:

    begin

    2.1. $P_0 := P_i$;

    2.2. $v := v_{i1}$;

    2.3. procedure A(P_0 , Π_0, P_0^0, N_2);

    2.4. Output (P_0, $\sum_{j=1}^{n_i} l_{ij}$) in X ;

    2.5. **For** $k = 2$ step 1 to $|P_0|$, **does** these steps:

    begin

    2.5.1. **If** $v \approx v_{ik}$ **then** go to the end of the cycle by k;

    else

    2.5.2. $P_0 := P_i \setminus v$; $\Pi_0 := \Pi_i$;

    2.5.3. L: = Function B (P_0, X);

    2.5.4. **If** L == true **then** do:

    begin;

    $M := \{\forall u \mid (u,v) \in P_0^1\}$;

    2.5.4.1. **If** K (G) == 1 **then** do

    begin;

    procedure A(P_0 , Π_0 ,M, N_1);

    output (Π_0, M) in X ;

    end;

    2.5.4.2. **else** do

    begin;

    procedure A (P_0 , Π_0 ,M, Σ_0);

    output (Π_0, M) in X ;

    end;

3. end;
4. end;

End of Algorithm 1.

Procedure A (G , Π , M, S) do the following:

// Must construct the embedding Π of a graph G (without vertices of degree 2) with a given number of vertices in the surface S (Euclidean plane, projective plane or Klein surface) and determine the cells on the boundaries of which are the set of vertices M //.

If a graph G has a subgraph or part of the graph H is homeomorphic K_5 or $K_{3,3}$, then we construct the embedding of these graphs in the projective plane, otherwise, we attach a graph to the Euclidean plane Σ_0. In nested graphs K_5 or $K_{3,3}$ a projective plane, there are cells s_5 , s_{33} with the following boundaries: ∂s_5 - a cycle of length 5 and 5 triangles for K_5 , or ∂s_{33} - a cycle of length 6 and 4 quadrilaterals for $K_{3,3}$, in which we will embed stars with centers taken from the subset $G^0 \setminus H^0$.

First of all, we will put all these stars in cells with either cycle boundaries of length 5 for or length 6 for and try to use no more than one additional Mobius strip glued to the cells ∂s_5 or ∂s_{33} . The number of vertices $|G^0|$ of the obstruction graph of the projective plane is at least 12. The number of options for the location of the centers and edges of stars, not more than 7 stars, is equal r^7 because each center of the star does not belong to two cells, where r the number of cells of the graph embedded in the projective plane $r = 6$ for K_5, $r = 5$ for $K_{3,3}$.

The time complexity of procedure A (G , Π , M, S) is proportional $O(r^7)$.

The function K (G) will determine the presence or absence of a graph G of a subgraph or part of a homeomorphic K_5 or $K_{3,3}$ and will give it out. To do this, we need to examine the complement of the $\overline{G}$ graph G for the presence of a subgraph of five isolated vertices, $\overline{K}_5$, or two triangles without common vertices, i.e. $2K_3$. If such subgraphs of the graph are detected, the function K (G) will give 1 and return to algorithm 1 the found vertices as vertices of the graph K_5 or $K_{3,3}$. In the absence case $\overline{K}_5$, $2K_3$ the function K (G) will give 0. The function B (P_0 , X) checks for the presence of an isomorphism of a graph P_0 with another element of the set of graphs X and will have polynomial complexity [10], [11] herein hand checking evidence identity of amalgamating sets of isomorphic graphs.

The part of the output result of algorithm 1 is on figures 3, 4.

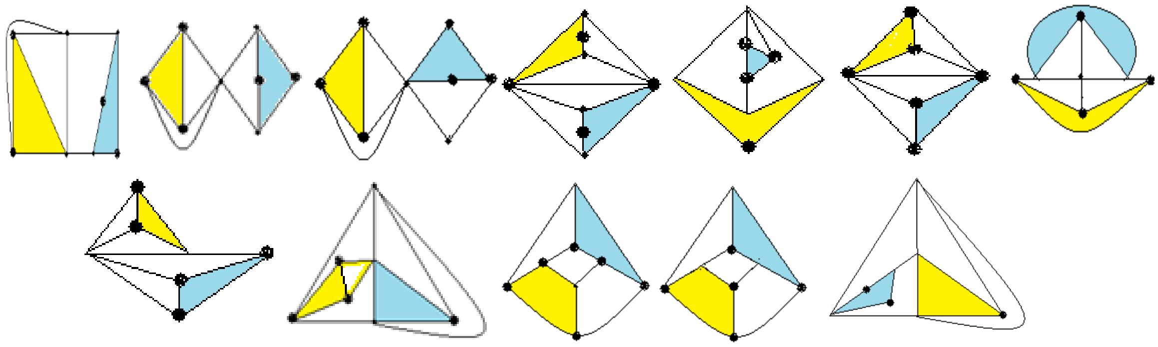
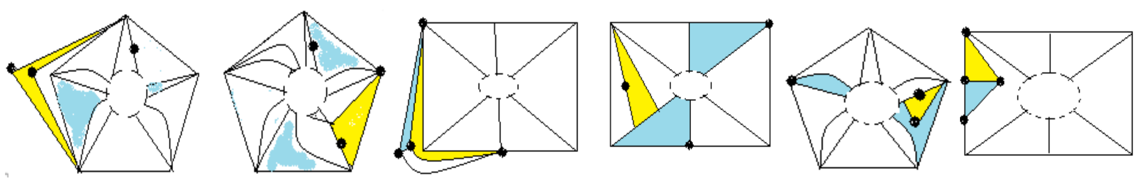


Fig. 3. Planar subgraphs of the projective minors with set M from push vertices at the boundaries of colored cells with the number of reachability 2 between two highlighted color 2-cells on the boundaries of which are subsets of the set M.



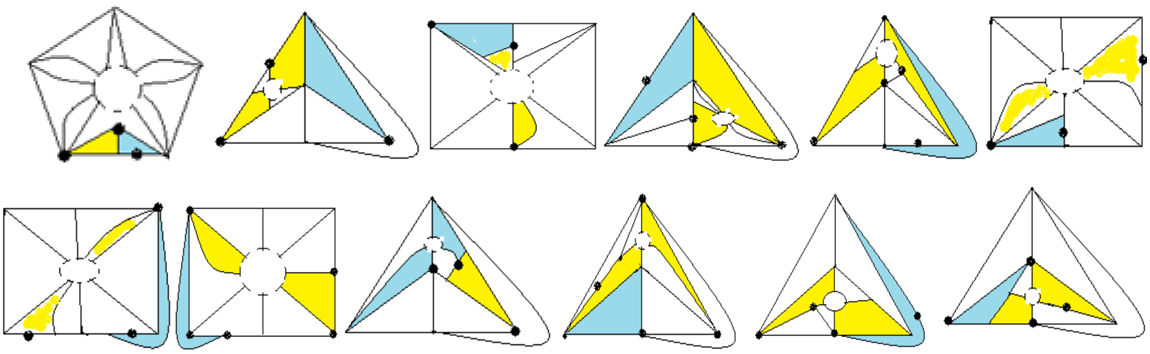


Fig. 4. The projective subgraphs of minors of the projective plane with two non-empty subsets of the set M, consisting of the push vertices and located on the bounderis of the colored 2-cells or 2- cell and pseudo-cell).

4. Acknowledgements

The authors gratefully acknowledge the support and help of officers of the organizing committee.

5. References

[1] M.P. Khomenko. φ -Transformation of Graphs. Institute of Mathematics, Kyiv, 1973.

[2] D. Archdeacon, C.P. Bonnington, N. Dean, N.Hartsfield. Obstructions Sets for Outer-Cylindrical Graphs. Journal of Graph Theory, v 38, i. 1, **2001, pp 42–64.**

[3] V. Petrenjuk. Characterization of the 3-minimal Planar Graph. Collection of the proceedings of a seminar of discrete mathematics\ and applications. Moscow, MGU 1993, p.217.

[4] V. Petrenjuk. List of 3-minimal Planar Graphs. Preprint DNTB 31.10.86 #2450-86. 7p.

[5] D. Archdeacon, N. Hartsfield, C. H. C. Little, B. Mohar. Obstructions sets for outer-projective-planar graphs. Ars Combinatoria 49, 1998, 113-128.

[6] Hur Surkhjin. The Kuratowski covering conjecture for graphs of the order less than 10. Dissertation, Ohio State University, 2008. URL: https://etd.ohiolink.edu/rws_etd/send_file /send?accession=osu1209141894&disposition=inline.

[7] B. Mohar, C. Thomassen. Graphs on surfaces, Johns Hopkins University Press, NY, 2001.

[8] A. Flòtotto. Embeddability of graphs into the Klein surface. Dissertation, Universitat Bielefeldvorgeleg, 2010. URL: https://d-nb.info/1007427973/34.

[9] V. Petrenjuk. About Transformation graphs as a tool for investigation. Proceedings of the 4-th International Conference on Computational Linguistics and Intelligent Systems (COLINS 2020). Volume I: Lviv, Ukraine, April 23-24, 2020, 1309-1319. URL: http://ceur-ws.org/Vol-2604/.

[10] LEDA: A library of efficient data types and algorithms, Max Planck Institute for Computer Science. URL: http://www.mpi-sb.mpg.de/LEDA.

[11] K. Scott. Outermobius and cylindrical graphs. Senior Thesis, Princeton University, 1997.

[12] V. Petrenjuk, D. Petrenjuk. List of Non-Outer Projective Planar Graphs. Proceedings of the Workshop Conference of COLINS-2021 the fifth edition of the International Conference on Computational Linguistics and Intelligent Systems, Kharkiv, Ukraine on April 22-23, 2021, Volume II, ISSN 2523-4013, 38-50pp. URL: https://web.kpi.kharkov.ua/iks/wp-content/uploads/sites/ 113/2021/10/CoLInS_Volume2_2021.pdf.