Property Analysis of Conditional Linear Random Process as a Mathematical Model of Cyclostationary Signal

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Abstract

The properties of probability characteristics of the conditional linear random process have been analyzed in the context of its application for the mathematical modelling of cyclostationary information signals. The class of conditional linear periodically correlated random processes has been characterized using the properties of their mathematical expectation and covariance function. The characteristic function method has been used to describe the class of conditional linear cyclostationary random processes represented in the form of the stochastic integral of cyclostationary kernel driven by the random process with independent periodic increments.

Keywords 1

Mathematical model, information signal, conditional linear cyclostationary random process, stochastic integral, kernel, independent increments, characteristic function, moment functions.

1. Introduction

Many information signals in technical, medical, or economical systems, the mathematical model of which can be represented in the form of a linear or conditional linear random process, also have the property of cyclostationarity, which can be caused by various factors, for example, daily, weekly, or seasonal cyclicity of electricity loads, gas or water consumption, the cyclicity of heart beats in the analysis of electro-cardio signals, the periodicity of photo stimulation in tasks of analysis of visually evoked biopotentials of the brain, etc.

The most general model of such kind of information signals is a cyclostationary random process whose finite-dimensional distribution functions (or characteristic functions) are periodic with respect to their time arguments [1, 2]. As the author [3] noted, the idea of stochastic periodicity belongs to E.E. Slutsky and is presented in his work "The Summation of Random Causes as the Source of Cyclic Processes". Therefore, in some works, cyclostationary processes are also called periodic by Slutsky [3]. A very important subclass of cyclostationary random processes is a periodically correlated random processes [3, 4], which have periodic moment functions of the first and second order.

A conditional linear random process (CLRP) is an important instrument for the problems of mathematical modelling of information signals that can be represented as a sum of large number of stochastically dependent random impulses occurring at Poisson times [5 - 7]. Thus, such kind of processes can be physically reasonable model of radar noise, many electrophysiological information signals, resource consumptions, vibration noises etc.

The constructive properties of linear and conditional linear random processes allow taking into account the causes of the rhythmic or cyclic properties of the studied information signals in the corresponding mathematical models [3, 6].

The main goal of the paper is to justify the conditions for CLRP to be periodically correlated random process, which is extension of the results of [6], and to describe the class of conditional linear cyclostationary random processes using the characteristic function expressions obtained in [7].

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2. Conditional Linear Periodically Correlated Random Processes

A real-valued conditional linear random process $\xi(\omega,t), \omega \in \Omega, t \in (-\infty,\infty)$ (where $\{\Omega,\mathfrak{F},\mathbf{P}\}$ is some probability space) is defined in the following form [6, 7]:

$$\xi(\omega,t) = \int_{-\infty}^{\infty} \varphi(\omega,\tau,t)d\eta(\omega,\tau), \omega \in \Omega, t \in \mathbb{R}, \quad (1)$$

where $\varphi(\omega,\tau,t), \tau,t \in \mathbb{R}$ is a real-valued *stochastic* kernel of CLRP; $\eta(\omega,\tau), \tau \in (-\infty,\infty)$ is a stochastically continuous Hilbert process with independent increments, satisfying the following conditions: $\mathbf{E}\eta(\omega,\tau) = a(\tau) < \infty$ and $\text{Var}[\eta(\omega,\tau)] = b(\tau) < \infty$, $\forall\tau$; random functions $\varphi(\omega,\tau,t)$ and $\eta(\omega,\tau)$ are *stochastically independent*.

The stochastic integral (1) is assumed to be exist in the mean-square convergence sense [6].

Mathematical expectation $\mathbf{E}\xi(\omega,t)$ and covariance function $R_\xi(t_1,t_2)$ of conditional linear random process (1) are represented as:

$$\mathbf{E}\xi(\omega,t) = \int_{-\infty}^{\infty} \phi(\tau,t)da(\tau), \quad (2)$$

$$R_\xi(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_\varphi(\tau_1,\tau_2;t_1,t_2)da(\tau_1)da(\tau_2) + \int_{-\infty}^{\infty} \mathbf{E}\big(\varphi(\omega,\tau,t_1)\varphi(\omega,\tau,t_2)\big)db(\tau). \quad (3)$$

where $\phi(\tau,t) = \mathbf{E}\varphi(\omega,\tau,t)$ is the mathematical expectation of the kernel of CLRP; $R_\varphi(\tau_1,\tau_2;t_1,t_2) = \mathbf{E}\big(\varphi_0(\omega,\tau_1,t_1)\varphi_0(\omega,\tau_2,t_2)\big)$ is the covariance function of the kernel of conditional linear random process ($\varphi_0(\omega,\tau,t) = \varphi(\omega,\tau,t) - \phi(\tau,t)$ is the centered kernel).

Let there exist the least real number (period) $T > 0$ and number $\alpha \in (-\infty,\infty)$ such that the process with independent increments $\eta(\omega,\tau)$ satisfies the following conditions:

$$da(\tau) = da(\tau + \alpha T) \text{ and } db(\tau) = db(\tau + \alpha T),$$

and mathematical expectation and covariance function of the kernel have the following properties:

$$\phi(\tau,t) = \phi(\tau + \alpha T, t + T), \quad (4)$$

$$R_\varphi(\tau_1,\tau_2;t_1,t_2) = R_\varphi(\tau_1 + \alpha T, \tau_2 + \alpha T; t_1 + T, t_2 + T). \quad (5)$$

then CLRP (1) is *periodically correlated random process*.

Indeed, mathematical expectation of CLRP in this case has the following property:

$$\mathbf{E}\xi(\omega,t) = \int_{-\infty}^{\infty} \phi(\tau,t)da(\tau) = \int_{-\infty}^{\infty} \phi(\tau + \alpha T, t + T)da(\tau + \alpha T) = \int_{-\infty}^{\infty} \phi(s,t+T)da(s) = \mathbf{E}\xi(\omega,t+T).$$

From (3), (4), and (5) it follows that the covariance function is expressed as:

$$R_\xi(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_\varphi(\tau_1 + \alpha T, \tau_2 + \alpha T; t_1 + T, t_2 + T)da(\tau_1 + \alpha T)da(\tau_2 + \alpha T) +$$

$$+ \int_{-\infty}^{\infty} R_\varphi(\tau + \alpha T, \tau + \alpha T; t_1 + T, t_2 + T)db(\tau + \alpha T) +$$

$$+ \int_{-\infty}^{\infty} \phi(\tau + \alpha T, t_1 + T)\phi(\tau + \alpha T, t_2 + T)db(\tau + \alpha T).$$

Let us denote: $\tau_1 + \alpha T = s_1$, $\tau_2 + \alpha T = s_2$, $\tau + \alpha T = s$, then we obtain:

$$R_\xi(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_\varphi(s_1,s_2;t_1 + T, t_2 + T)da(s_1)da(s_2) +$$

$$+ \int_{-\infty}^{\infty} R_\varphi(s,s;t_1 + T, t_2 + T)db(s) + \int_{-\infty}^{\infty} \phi(s,t_1 + T)\phi(s,t_2 + T)db(s) =$$

$$= R_\xi(t_1 + T, t_2 + T).$$

The mathematical expectation and the covariance function of the considered process are periodic with respect to their arguments. Thus, it is a *conditional linear periodically correlated random process*.

3. Conditional Linear Cyclostationary Random Processes

Let $\mathfrak{F}_\varphi \subset \mathfrak{F}$ be a σ-subalgebra generated by the random kernel $\varphi(\omega,\tau,t)$ satisfying the following conditions [7]:

$$\int_{-\infty}^{\infty} |\varphi(\omega,\tau,t)||da(\tau)| < \infty, \quad \int_{-\infty}^{\infty} |\varphi(\omega,\tau,t)|^2 \, db(\tau) < \infty, \ \forall t \text{ with probability 1.}$$

The m-dimensional characteristic function of CLRP is represented as [7]:

$$f_\xi(u_1,u_2,...,u_m;t_1,t_2,...,t_m) = \mathbf{E}\exp\left[i\sum_{k=1}^{m} u_k \xi(\omega,t_k) \right] = \mathbf{E} f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m),$$

where $f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m) = \mathbf{E}\left(\exp\left[i\sum_{k=1}^{m} u_k \xi(\omega,t_k) \right] \middle| \mathfrak{F}_\varphi \right)$ is conditional (with respect to $\mathfrak{F}_\varphi$) characteristic function ($\mathfrak{F}_\varphi$-characteristic function) of CLRP (1), which is expressed as follows [7]:

$$f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m) = \exp\left[i\sum_{k=1}^{m} u_k \int_{-\infty}^{\infty} \varphi(\omega,\tau,t_k) da(\tau) + \right.$$
$$\left. + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\exp\left[ix\sum_{k=1}^{m} u_k \varphi(\omega,\tau,t_k) \right] - 1 - ix\sum_{k=1}^{m} u_k \varphi(\omega,\tau,t_k) \right) \frac{d_x d_\tau K(x;\tau)}{x^2} \right], \quad (6)$$
$$u_k, t_k \in (-\infty,\infty), \ k = \overline{1,m},$$

where $K(x;\tau)$ is the Poisson jump spectrum in Kolmogorov's form of infinitely divisible random process with independent increments $\eta(\omega,\tau)$.

Let us now consider the general case of conditional linear cyclostationary random process. Namely, using the method of characteristic functions, we prove the sufficient conditions (which are important for applications) that the kernel $\varphi(\omega,\tau,t)$ and the process $\eta(\omega,\tau)$ have to satisfy in order for the conditional linear random process to be cyclostationary.

Let there exist the least real number (period) $T > 0$ and number $\alpha \in (-\infty,\infty)$ such that:

- random functions (fields) $\varphi(\omega,\tau,t)$ and $\varphi(\omega,\tau+\alpha T,t+T)$ are stochastically equivalent in the wide sense, that is, their finite-dimensional distributions are equal:

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m} \{\omega : \varphi(\omega,\tau_i,t_j) < x_{ij}\} \right) = \mathbf{P}\left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m} \{\omega : \varphi(\omega,\tau_i+\alpha T,t_j+T) < x_{ij}\} \right), x_{ij} \in \mathbb{R}; \quad (7)$$

- $\eta(\omega,\tau)$ is a random process with independent increments satisfying the following properties:

$$da(\tau) = da(\tau+\alpha T),$$
$$d_x d_\tau K(x;\tau) = d_x d_\tau K(x;\tau+\alpha T). \quad (8)$$

Then the characteristic function of CLRP (1) is T-periodic by its time arguments, that is

$$f_\xi(u_1,u_2,...,u_m;t_1,t_2,...,t_m) = f_\xi(u_1,u_2,...,u_m;t_1+T,t_2+T,...,t_m+T). \quad (9)$$

Thus, the process (1), satisfying (7) and (8) is *conditional linear cyclostationary random process*.

Note that, the process with independent increments $\eta(\omega,\tau)$ in the above statement is the random process with independent αT-periodical increments [3]. That is, the characteristic function $f_{\Delta\eta}(u;s,\tau) = \mathbf{M}e^{iu\Delta^s\eta(\omega,\tau)}$ of the increments $\Delta^s\eta(\omega,\tau) = \eta(\omega,\tau) - \eta(\omega,s)$, $s<\tau$ of such kind of process $\eta(\omega,\tau)$ satisfies the following condition:

$$f_{\Delta\eta}(u;s,\tau) = f_{\Delta\eta}(u;s+\alpha T,\tau+\alpha T).$$

Parameter $\alpha \in (-\infty,\infty)$ is a ratio of the period of increments of the random process $\eta(\omega,\tau)$ and period of the random process $\xi(\omega,t)$ (see also the same property in [3]). We should note, that in many problems of mathematical modelling of information signals in technical, medical, or economical applications $\alpha = 1$, that is, period of increments of the process $\eta(\omega,\tau)$ is equal to the period of the kernel $\varphi(\omega,\tau,t)$ in the sense of (7) and (8).

We can prove the above statement analyzing the properties of m-dimensional characteristic function of CLRP taking into account the conditions (7) and (8).

Like in [7] we further write $\text{Law}(\xi(\omega)) = \text{Law}(\eta(\omega))$ if random variables $\xi(\omega)$ and $\eta(\omega)$ have the same distribution functions (distribution laws). So, we can write the following:

$$\text{Law}\left(\sum_{k=1}^{m} u_k \int_{-\infty}^{\infty} \varphi(\omega,\tau,t_k)da(\tau)\right) = \text{Law}\left(\sum_{k=1}^{m} u_k \int_{-\infty}^{\infty} \varphi(\omega,\tau+\alpha T,t_k+t)da(\tau+\alpha T)\right) =$$

$$= \text{Law}\left(\sum_{k=1}^{m} u_k \int_{-\infty}^{\infty} \varphi(\omega,s,t_k+t)da(s)\right),$$

$$\text{Law}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{ix\sum_{k=1}^{m}u_k\varphi(\omega,\tau,t_k)} - 1 - ix\sum_{k=1}^{m}u_k\varphi(\omega,\tau,t_k)\right)\frac{d_x d_\tau K(x;\tau)}{x^2}\right) =$$

$$= \text{Law}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{ix\sum_{k=1}^{m}u_k\varphi(\omega,\tau+\alpha T,t_k+T)} - 1 - ix\sum_{k=1}^{m}u_k\varphi(\omega,\tau+\alpha T,t_k+T)\right)\frac{d_x d_\tau K(x;\tau+\alpha T)}{x^2}\right) =$$

$$= \text{Law}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{ix\sum_{k=1}^{m}u_k\varphi(\omega,s,t_k+T)} - 1 - ix\sum_{k=1}^{m}u_k\varphi(\omega,s,t_k+T)\right)\frac{d_x d_\tau K(x;s)}{x^2}\right),$$

where $s = \tau + \alpha T$.

From the above expressions we can conclude that the probability distribution of random m-dimensional $\mathfrak{F}_\varphi$-characteristic function of conditional linear random process has the following property:

$$\text{Law}(f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m)) = \text{Law}(f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1+T,t_2+T,...,t_m+T)).$$

Taking into account the $f_\xi(u_1,u_2,...,u_m;t_1,t_2,...,t_m) = \mathbf{E}f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m)$, we can conclude that characteristic function of the CLRP satisfies (9).

4. Discussion

The obtained results can be used for mathematical model identification of information signals which are physically generated as a sum of a large amount of stochastically dependent impulses occurring at random Poisson times, that is when the model of conditional linear random process is applicable. Comparing with the results of [7] (where conditions have been proven of CLRP to be stationary) the sufficient conditions of cyclostationarity of such kind of models have been proven in the paper.

In papers [5-7] the connection of CLRP with models of stochastic linear dynamic systems is noted. Their identification is carried out on the basis of observation and analysis of signals at the input and output of the systems. For the CLRP model, obviously, the generating process with independent increments is not available for observation. Identification is carried out on the basis of a theoretical analysis of the characteristics of CLRP, as well as their statistical estimation based on the results of registration of discrete time information signals.

However, the CLRP model with continuous time (including the cyclostationary case) makes it possible to take into account some important a priori information about the research object and the corresponding stochastic information signal. In particular, identification of the class of the model (stationary process or cyclostationary, etc.), formulation of hypotheses regarding the distribution of the generating process, its homogeneity, justification of the properties of the model kernel, etc. can be carried out on the basis of the analysis of the physical (biophysical, economic, etc.) mechanism of generation of the simulated information signal. Thus, the identified model with continuous time becomes adequate for the studied information signal, because it is based on its physical nature. It is clear that theoretical conclusions or hypotheses made on the basis of the model constructed in this way can be further confirmed on the basis of experimental data.

The structure of identification of information signal mathematical models based on conditional linear cyclostationary random process has been represented on the Figure 1.

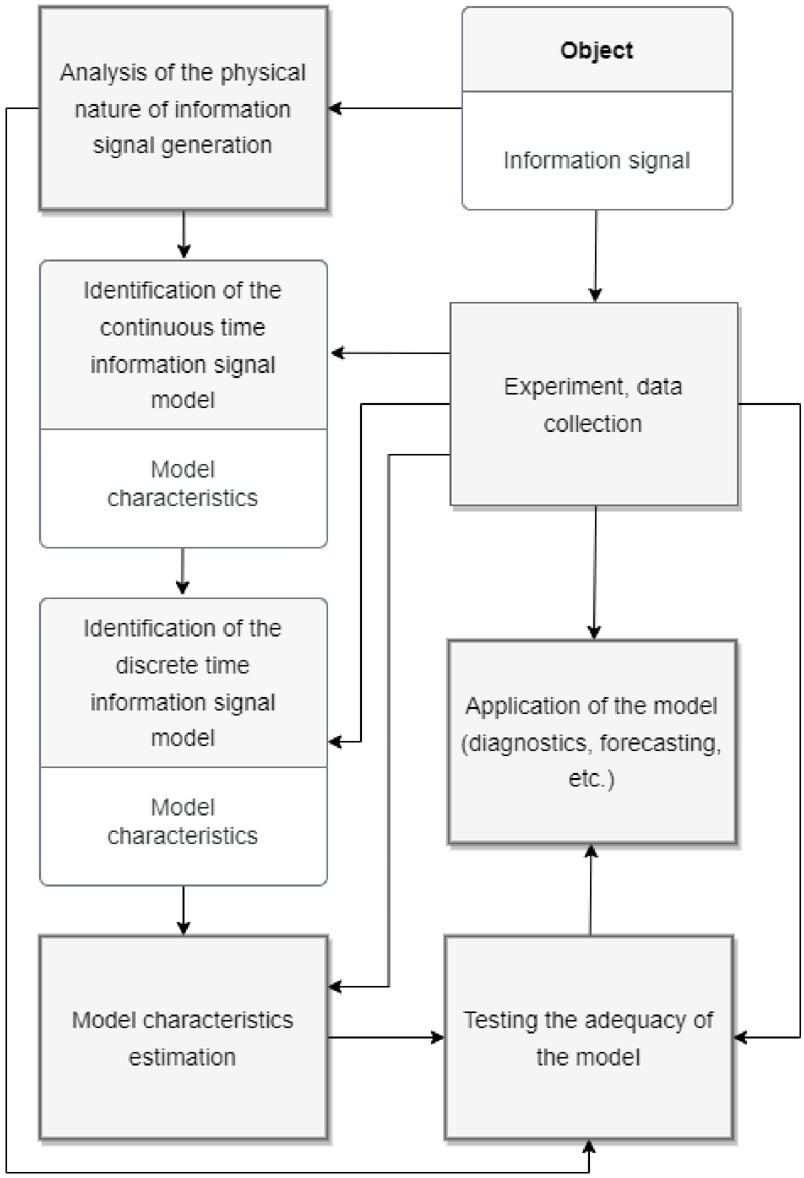

Figure 1: Structure of identification of information signal mathematical models based on conditional linear cyclostationary random process

We can see that the identification of the appropriate model with discrete time, as well as its informative characteristics detection, the justification of the methods of their statistical estimation is carried out on the basis of the primary model with continuous time.

In the context of model characteristics estimation, we should emphasize the great role of periodic autoregressive and moving average methods [8, 9] for the cyclostationary linear model identification and informative features selection. The corresponding approach for the case of CLRP should consists of the following. The discrete-time conditional linear cyclostationary random process can be interpreted as the response of a digital filter with random cyclostationary parameters to the input cyclostationary white noise. If this filter is built so that it has only a recursive structure, then the random signal at its output will be a cyclostationary autoregressive process with random coefficients [10]. So, application of the random coefficient periodic autoregressive methods for conditional linear cyclostationary random process identification is a prospective task.

5. Conclusion

The class of conditional linear periodically correlated random processes has been characterized. Each element of the class is the conditional linear random process driven by the process with independent periodic increments, the random kernel of the process has periodic mathematical expectation, the covariance function of the kernel is periodic by its time arguments with the same period.

The class of conditional linear cyclostationary random processes has been characterized using the characteristic functions method. Each element of the class is the conditional linear random process driven by the process with independent periodic increments, the random kernel of the process is the cyclostationary bivariate random field. Also, the considered processes are the mixtures of infinitely divisible distributions.

The general approach for applied identification of information signal mathematical models based on conditional linear cyclostationary random process has been analyzed based on the above theoretical results.

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