Ergodicity and Mixing of Conditional Linear Random Processes in the Problems of Information Signal Modelling and Analysis

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Abstract

Ergodicity is a fundamental property in statistical information signal modelling and processing because it simplifies the analysis and estimation of signal properties and system parameters. It enables practitioners to work with single realizations of signals and make meaningful statistical inferences, which is necessary when dealing with real-world data and signals. The continuous-time stationary conditional linear random process as a mathematical model of information signals has been analyzed in the paper using characteristic functions method. The mixing property of the process, from which its ergodicity follows, has been proven.

Keywords

Mathematical model, information signal, conditional linear random process, ergodicity, mixing, kernel, Levy process, characteristic function, moment functions

1. Introduction

Conditional linear random process (CLRP) is represented as a stochastic integral with a random kernel driven by the process with independent increments [1, 2]. It is used for the mathematical modelling, computer simulation, statistical analysis and forecasting of information signals and processes which can be represented as a sum of many random stochastically dependent impulses occurring at Poisson time moments. The CLRP as a mathematical model of the investigated signal considers the physical nature of its generation. The CLRP is applied in the area of information systems and technology for the problems of mathematical modelling and analysis of electrophysiological information signals, radar clutter, dynamic loads of mechanical systems, forecasting of energy loads and consumptions, water consumptions, etc. [2–4]. The conditional linear random process is a generalization of a well-known model of the linear random process [5–7] having a similar integral representation but with a nonrandom kernel, and as a result they can be used only for mathematical modelling of the signals or processes represented as a sum of independent impulses. Conditional linear random processes compared with their linear counterparts take into account the conditional heteroscedasticity of modelled signals which is important for information technology applications in economics, medicine, and energy.

Ergodicity is always the important property of mathematical models which is used for information signal processing when the task is to estimate parameters of a signal or a system [8–10]. Ergodicity allows to use of time averages (averages over a single realization of a signal) to estimate these parameters. This is particularly important when dealing with non-stationary signals or time-varying systems [11, 12]. Ergodicity is closely related to the concept of stationarity. The assumption of ergodicity is fundamental in the modelling of communication systems and the analysis of random noise in electrical circuits, application to financial mathematics [13], compressive sensing [14, 15], etc.

Proceedings ITTAP'2023: 3rd International Workshop on Information Technologies: Theoretical and Applied Problems, November 22–24, 2023, Ternopil, Ukraine, Opole, Poland
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Very often the ergodic property of the investigated signal is just a hypothesis or assumption. However it is a characteristic property of linear random processes [16, 17]. Moreover, the mixing property is inherent in linear random processes [17]. We did not find such properties of conditional linear random processes in the available literature. Thus, it is important to study the same features for the CLRP model.

The main goal of the paper is to justify the conditions for the continuous-time stationary conditional linear random process to be ergodic as a consequence of mixing property using the known representation of its multidimensional characteristic function.

In the following parts of the article, we define the continuous-time conditional linear random process and represent its multidimensional characteristic function. Then we consider the notions of ergodicity and mixing in terms of characteristic functions of the stationary random processes. Finally, we find the conditions of stationary CLRP to be mixing and ergodic.

2. Conditional linear random process and its properties

This section is preliminary and covers the definition of continuous-time conditional linear random process and representation of its multidimensional characteristic function which is used in the next section for proving the mixing property and ergodicity. Here we follow mostly the results of [1] and [2], but Levy process is used with its Poisson jump spectrum given on the Levy-Khintchine form in the CLRP model definition.

A continuous-time conditional linear random process $\xi(\omega,t), \omega\in\Omega, t\in(-\infty,\infty)$ (where $\{\Omega,\mathfrak{F},\mathbf{P}\}$ is probability space) is defined as the following stochastic integral:

$$\xi(\omega,t)=\int_{-\infty}^{\infty}\varphi(\omega,\tau,t)d\eta(\omega,\tau),\ \omega\in\Omega,\ t\in\mathbb{R}, \tag{1}$$

where $\varphi(\omega,\tau,t),\ \tau,t\in\mathbb{R}$ is a *stochastic* kernel;
$\eta(\omega,\tau),\ \tau\in(-\infty,\infty)$ is a Levy process;
random functions $\varphi(\omega,\tau,t)$ and $\eta(\omega,\tau)$ are *stochastically independent*.

Let $\mathfrak{F}_\varphi\subset\mathfrak{F}$ be a σ-subalgebra generated by the random kernel $\varphi(\omega,\tau,t)$ satisfying the condition $\int_{-\infty}^{\infty}\left|\varphi(\omega,\tau,t)\right|<\infty$ with probability 1.

The m-dimensional characteristic function of conditional linear random process (1) is represented in the following form:

$$f_\xi(u_1,u_2,...,u_m;t_1,t_2,...,t_m)=\mathbf{E}\exp\left[i\sum_{k=1}^{m}u_k\xi(\omega,t_k)\right]=\mathbf{E}f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m),$$

where $f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m)=\mathbf{E}\left(\exp\left[i\sum_{k=1}^{m}u_k\xi(\omega,t_k)\right]\middle|\mathfrak{F}_\varphi\right)$ is conditional (with respect to $\mathfrak{F}_\varphi$) characteristic function ($\mathfrak{F}_\varphi$-characteristic function) of CLRP (1), which is expressed as follows:

$$f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m)=\exp\left[ia\left(\sum_{k=1}^{m}u_k\int_{-\infty}^{\infty}\varphi(\omega,\tau,t_k)d\tau+\right.\right.$$

$$\left.+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\exp\left[ix\left(\sum_{k=1}^{m}u_k\varphi(\omega,\tau,t_k)\right)\right]-1-\frac{ix\left(\sum_{k=1}^{m}u_k\varphi(\omega,\tau,t_k)\right)}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x)d\tau\right], \tag{2}$$

$$u_k,t_k\in(-\infty,\infty),\ k=\overline{1,m},$$

where $G(x), x\in\mathbb{R}$ is a real non-decreasing and bounded function such that $G(-\infty)=0$ (the Poisson jump spectrum in Levy-Khintchine form of infinitely divisible Levy process $\eta(\omega,\tau)$);

$a \in \mathbb{R}$, and if $\mathbf{E}\eta(\omega,\tau) < \infty$ then $a = \mathbf{E}\eta(\omega,\tau) - \int_{-\infty}^{\infty} x dG(x).$

If random functions (fields) $\varphi(\omega,\tau,t)$ and $\varphi(\omega,\tau+s,t+s)$ are stochastically equivalent in the wide sense, that is, their finite-dimensional distributions are equal satisfying the following condition:

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\left\{\omega : \varphi(\omega,\tau_i,t_j) < x_{ij}\right\}\right) = \mathbf{P}\left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\left\{\omega : \varphi(\omega,\tau_i+s,t_j+s) < x_{ij}\right\}\right), x_{ij} \in \mathbb{R}; \quad (3)$$

for any $s \in \mathbb{R}$, then conditional linear random process (1) is a strict sense stationary.

Using the above representation of m-dimensional characteristic function of conditional linear random process the expressions for moment functions can be obtained which are important for information signal processing (including mathematical expectation and covariance function), properties of cyclostationarity [18] can be analyzed, mixing and ergodicity conditions can be proven. The results can be also extended for multivariate case.

3. Ergodicity and mixing of stationary conditional linear random proces

In this section we define the general notion of continuous-time stationary ergodic random process. We also analyze some cases which are important for the problems of information signal processing. Finally, we define and prove the mixing property for conditional linear random process. Ergodicity is the consequence of mixing property [17, 19].

Let $\xi(\omega,t), t \in (-\infty,\infty)$ is a continuous-time strictly stationary random process with the values in a measurable space $\{X,\mathfrak{B}\}$ and $g(x_1,x_2,...,x_m)$, $m \geq 1$ is a $\mathfrak{B}^{m}$-measurable function satisfying the following condition:

$$\mathbf{E}g\left(\xi(\omega,t_1),\xi(\omega,t_2),...,\xi(\omega,t_m)\right) < \infty, \forall t_1,t_2,...,t_m \in \mathbb{R}. \quad (4)$$

The continuous-time strictly stationary random process $\xi(\omega,t), t \in (-\infty,\infty)$ is called *ergodic* if for any function $g(x_1,x_2,...,x_m)$ satisfying the above conditions, the following holds with probability 1:

$$\lim_{c\to\infty}\frac{1}{c}\int_0^c g\left(\xi(\omega,t_1+t),\xi(\omega,t_2+t),...,\xi(\omega,t_m+t)\right)dt = \\ = \mathbf{E}g\left(\xi(\omega,t_1),\xi(\omega,t_2),...,\xi(\omega,t_m)\right), \forall t_1,t_2,...,t_m \in \mathbb{R}. \quad (5)$$

The above definition of ergodic random process is very general. But in the problems of information signal analysis several cases are the most important. The mathematical expectation and covariance function, one- and two-dimensional cumulative distribution functions and characteristic functions of the investigated signal are usually estimated, analyzed, and used for informative features detection [20]. Further, the ergodicity of stationary random process with respect to the most important probability characteristics are represented indicating the relation to the general condition (5).

On the below expressions m is the dimension of the corresponding function $g(x_1,x_2,...,x_m)$, convergence is assumed with probability 1. Thus, we can define the following types of ergodicity of continuous-time strictly stationary random process $\xi(\omega,t), t \in (-\infty,\infty)$ [17].

Ergodicity with respect to the mathematical expectation $\mathbf{E}\xi(\omega,t) = \mu$:

$$m=1,\ g(x)=x,\ t_1=0 \Rightarrow \lim_{c\to\infty}\frac{1}{c}\int_0^c \xi(\omega,t)dt = \mu.$$

Ergodicity with respect to the covariance function $R(\tau) = \mathbf{E}\left[\left(\xi(\omega,t)-\mu\right)\left(\xi(\omega,t+\tau)-\mu\right)\right]$, $\tau \in \mathbb{R}$:

$$m=2,\ g(x_1,x_2)=(x_1-\mu)(x_2-\mu),\ t_1=0,\ t_2=\tau \Rightarrow \lim_{c\to\infty}\frac{1}{c}\int_0^c \left(\xi(\omega,t)-\mu\right)\left(\xi(\omega,t+\tau)-\mu\right)dt = R(\tau).$$

Ergodicity with respect to the one-dimensional cumulative distribution function $F_{\xi}(y) = \mathbf{P}(\xi(\omega,t) < y), y \in \mathbb{R}$:

$$m = 1, g(x) = U(y - x), t_1 = 0 \Rightarrow \lim_{c\to\infty}\frac{1}{c}\int_0^c U(y - \xi(\omega,t)dt = F_\xi(y),$$

where $U(y) = \begin{cases} 0, y \le 0 \\ 1, y > 0 \end{cases}$ is a Heaviside function.

Ergodicity with respect to the two-dimensional cumulative distribution function $F_\xi(y_1, y_2; \tau) = \mathbf{P}(\xi(\omega,t) < y_1, \xi(\omega,t+\tau) < y_2),\ y_1, y_2 \in \mathbb{R}$:

$$m = 2,\ \ g(x_1, x_2) = U(y_1 - x_1)U(y_2 - x_2),\ t_1 = 0,\ t_2 = \tau \Rightarrow$$

$$\Rightarrow \lim_{c\to\infty}\frac{1}{c}\int_0^c U(y_1 - \xi(\omega,t)U(y_2 - \xi(\omega,t+\tau)dt = F_\xi(y_1, y_2; \tau).$$

Ergodicity with respect to the one-dimensional characteristic function $f_\xi(u) = \mathbf{E}\exp\left(iu\xi(\omega,t)\right)$:

$$m = 1, g(x) = \exp\left(iux\right),\ t_1 = 0 \Rightarrow \lim_{c\to\infty}\frac{1}{c}\int_0^c \exp(iu\xi(\omega,t))dt = f_\xi(u).$$

Ergodicity with respect to the two-dimensional characteristic function $f_\xi(u_1, u_2; \tau) =$ $= \mathbf{E}\exp\left[i\left(u_1\xi(\omega,t) + u_2\xi(\omega,t+\tau)\right)\right]$:

$$m = 2,\ \ g(x_1, x_2) = \exp\left[i(u_1x_1 + u_2x_2)\right],\ t_1 = 0,\ t_2 = \tau \Rightarrow$$

$$\Rightarrow \lim_{c\to\infty}\frac{1}{c}\int_0^c \exp\left[i\left(u_1\xi(\omega,t) + u_2\xi(\omega,t+\tau)\right)\right]dt = f_\xi(u_1, u_2; \tau).$$

It is easy to see that stationary random process can be ergodic with respect to mathematical expectation but not ergodic with respect to covariance function or characteristic function, etc.

The notion of *ergodicity* as it has been defined above represents the convergence of corresponding sampling averages to probability characteristics of the random process. *Mixing* is another fundamental property of random process expressed the fact that events (related to investigated process) separated by long time intervals are approximately independent [19].

In terms of random process distributions, mixing property means that random vectors $(\xi(\omega,t_1+t), \xi(\omega,t_2+t), ..., \xi(\omega,t_m+t))$ and $(\xi(\omega,s_1), \xi(\omega,s_2), ..., \xi(\omega,s_n))$ (created by the samples of stationary random process) become approximately independent as $|t| \to \infty$.

A mixing property of the stationary random process represented by its characteristic functions is defined as follows [17]:

$$\begin{gathered} \lim_{|t|\to\infty} f_\xi(u_1, u_2, ..., u_m, v_1, v_2, ..., v_n;\ t_1+t, t_2+t, ..., t_m+t, s_1, s_2, ..., s_n) = \\ = f_\xi(u_1, u_2, ..., u_m; t_1, t_2, ..., t_m) f_\xi(v_1, v_2, ..., v_n; s_1, s_2, ..., s_n), \\ u_1, u_2, ..., u_m, v_1, v_2, ..., v_n \in \mathbb{R},\quad t_1, t_2, ..., t_m, s_1, s_2, ..., s_n \in \mathbb{R}. \end{gathered} \tag{6}$$

Ergodicity in the sense of (5) is a consequence of mixing. The hierarchy of mixing and ergodicity properties which are considered in the paper has been illustrated in Figure 1. Thus, to justify the ergodic properties of continuous-time strictly stationary conditional linear random process we need to prove the mixing property first. The idea is to use the characteristic function method using the representation (2) and relationship between conditional and unconditional characteristic functions. Obviously, the time-dependent properties of the model expressed as stochastic integral driven by Levy process heavily depend on the corresponding properties of the random kernel.

Let $\xi(\omega,t), t \in (-\infty,\infty)$ be a continuous-time strictly stationary conditional linear random process driven by Levy process, and with the kernel satisfying (3).

Let us denote

$$\mathrm{Law}(\xi_1(\omega), \xi_2(\omega), ..., \xi_m(\omega)) = \mathrm{Law}(\eta_1(\omega), \eta_2(\omega), ..., \eta_m(\omega))$$

if random vectors

$$(\xi_1(\omega), \xi_2(\omega), ..., \xi_m(\omega)) \text{ and } (\eta_1(\omega), \eta_2(\omega), ..., \eta_m(\omega))$$

have the same m-dimensional cumulative distribution functions (distribution laws).

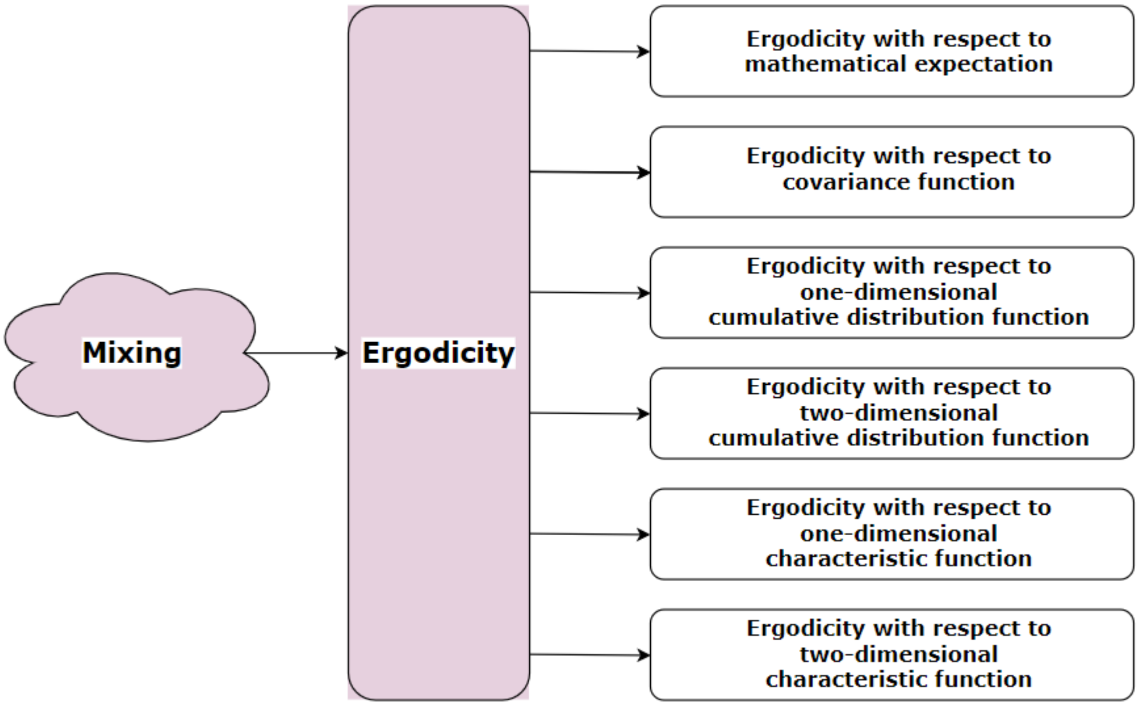

Figure 1: Hierarchy of mixing and ergodicity properties

Let random vectors

$$(\varphi(\omega,\tau,t_1+t),\varphi(\omega,\tau,t_2+t),\dots,\varphi(\omega,\tau,t_m+t)) \text{ and } (\varphi(\omega,\tau,s_1),\varphi(\omega,\tau,s_2),\dots,\varphi(\omega,\tau,s_n))$$

are asymptotically independent as $|t|\to\infty$, $\forall\tau$, $t_1,t_2,\dots,t_m,s_1,s_2,\dots,s_n\in\mathbb{R}$, that is, taking into account (3), the following condition holds:

$$\lim_{|t|\to\infty}\mathrm{Law}\Big(\varphi(\omega,\tau+t,t_1+t),\varphi(\omega,\tau+t,t_2+t),\dots,\varphi(\omega,\tau+t,t_m+t),\varphi(\omega,\tau,s_1),\varphi(\omega,\tau,s_2),\dots,\varphi(\omega,\tau,s_n)\Big)=$$
$$=\mathrm{Law}\big(\varphi(\omega,\tau,t_1),\varphi(\omega,\tau,t_2),\dots,\varphi(\omega,\tau,t_m)\big)\mathrm{Law}\big(\varphi(\omega,\tau,s_1),\varphi(\omega,\tau,s_2),\dots,\varphi(\omega,\tau,s_n)\big). \quad (7)$$

Then $\xi(\omega,t)$ is the strictly stationary CLRP satisfying the mixing condition (6) $\forall t_1,t_2,\dots,t_m,s_1,s_2,\dots,s_n\in\mathbb{R}$.

Indeed, $(m+n)$-dimensional $\mathfrak{F}_\varphi$-characteristic function of the stationary CLRP with probability 1 can be represented as:

$$f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,\dots,u_m,v_1,v_2,\dots,v_n;t_1+t,t_2+t,\dots,t_m+t,s_1,s_2,\dots,s_n)=$$
$$=\mathbf{E}\left(\exp\left(i\sum_{k=1}^{m}u_k\xi(\omega,t_k+t)+i\sum_{k=1}^{n}v_k\xi(\omega,s_k)\right)\middle|\mathfrak{F}_\varphi\right)=$$
$$=\exp\left[ia\left(\sum_{k=1}^{m}u_k\int_{-\infty}^{\infty}\varphi(\omega,\tau,t_k+t)d\tau+\sum_{k=1}^{n}v_k\int_{-\infty}^{\infty}\varphi(\omega,\tau,s_k)d\tau\right)+\right.$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\exp\left(ix\left(\sum_{k=1}^{m}u_k\varphi(\omega,\tau,t_k+t)+\sum_{k=1}^{n}v_k\varphi(\omega,\tau,s_k)\right)\right)-1-\frac{ix\left(\sum_{k=1}^{m}u_k\varphi(\omega,\tau,t_k+t)+\sum_{k=1}^{n}v_k\varphi(\omega,\tau,s_k)\right)}{1+x^2}\right)\times$$
$$\left.\times\frac{1+x^2}{x^2}dG(x)d\tau\right]=$$

$$= \exp\left[ia\left(\sum_{k=1}^{m} u_k \int_{-\infty}^{\infty} \varphi(\omega,\tau,t_k+t)d\tau + \sum_{k=1}^{n} v_k \int_{-\infty}^{\infty} \varphi(\omega,\tau,s_k)d\tau + \right.\right.$$

$$\left.\left. + \int_{-\infty}^{\infty} \psi\left(\sum_{k=1}^{m} u_k \varphi(\omega,\tau,t_k+t) + \sum_{k=1}^{n} v_k \varphi(\omega,\tau,s_k) \right) d\tau \right],$$

where

$$\psi(u) = \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x)$$

is the logarithm of characteristic function of infinitely divisible random variable $\eta(\omega,1)$ on the Levy-Khintchine form when $a = 0$.

The function $\psi(u)$ is a uniformly continuous on $u \in \mathbb{R}$ and $\psi(0) = 0$.

Since a kernel of CLRP satisfies the condition $\int_{-\infty}^{\infty} |\varphi(\omega,\tau,t)| d\tau < \infty$, $\forall t \in \mathbb{R}$ with probability 1, then $\varphi(\omega,\tau,t) \to 0$ as $|\tau| \to \infty$, $\forall t \in \mathbb{R}$ with probability 1. Taking into account (3) and (7), properties of the above function $\psi(u)$, and also properties of $\mathfrak{F}_\varphi$-characteristic function [2], the following expressions with respect to conditional and unconditional characteristic functions of CLRP can be written:

$$\lim_{|t|\to\infty} f_\xi(u_1,u_2,...,u_m,v_1,v_2,...,v_n;t_1+t,t_2+t,...,t_m+t,s_1,s_2,...,s_n) =$$

$$= \lim_{|t|\to\infty} \mathbf{E} f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m,v_1,v_2,...,v_n;t_1+t,t_2+t,...,t_m+t,s_1,s_2,...,s_n) =$$

$$= \mathbf{E} \lim_{|t|\to\infty} f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m,v_1,v_2,...,v_n;t_1+t,t_2+t,...,t_m+t,s_1,s_2,...,s_n) =$$

$$= \mathbf{E}\left(f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m) f_\xi^{\mathfrak{F}_\varphi}(\omega,v_1,v_2,...,v_n;s_1,s_2,...,s_n) \right) =$$

$$= \mathbf{E} f_\xi^{\mathfrak{F}_\varphi}(\omega,u_1,u_2,...,u_m;t_1,t_2,...,t_m) \mathbf{E} f_\xi^{\mathfrak{F}_\varphi}(\omega,v_1,v_2,...,v_n;s_1,s_2,...,s_n) =$$

$$= f_\xi(u_1,u_2,...,u_m;t_1,t_2,...,t_m) f_\xi(v_1,v_2,...,v_n;s_1,s_2,...,s_n).$$

Thus, the condition (6) holds. That is, the investigated stationary conditional linear random process undergoes the mixing property. The ergodicity of the process in the sense of (5) is an immediate corollary of mixing property, including ergodicity with respect to moment functions, one- and multidimensional cumulative distribution functions and characteristic functions which are used for the feature detection and estimation in the problems of information signal processing.

4. Conclusions

The conditional linear random process driven by Levy process has been defined and described using conditional characteristic functions method, Levy-Khintchine form has been used to specify the Poisson jump spectrum of Levy process. The CLRP belongs to the class of infinitely divisible mixtures. The condition of CLRP to be strict sense stationary has been represented.

The hierarchy of mixing and ergodicity properties has been analyzed, the importance of the corresponding concepts for the problems of applied information signal modelling and processing has been represented. Thus, the effective mathematical model should undergo ergodicity and mixing properties.

The mixing property of the continuous-time strictly stationary conditional linear random process has been proven using the characteristic functions method. The ergodicity property is a consequence of mixing. That is why, if the information signal is modelled as stationary ergodic CLRP then it is priori justification of performing the statistical analysis using time averaging of its single realization.

The prospective research is related to the study of mixing and ergodicity of discrete-time conditional linear random process.

5. References

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