Weak Chaos and Controllability Conditions in Discrete Systems

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Abstract
Report is devoted to obtaining the conditions of the input-to-state stability for the discrete linear systems in the Banach and Hilbert spaces. For the nonlinear boundary-value problem we found a necessary condition for the existence of bounded solutions. Such condition was obtained with using of the system of operator equations for generating elements. Moreover, conditions of the controllability and reachability were obtained with using of the operator matrix equation. Estimates on the norms of solutions were obtained under assumption that the corresponding linear interconnected system admits a discrete dichotomy. For the boundary-value problem conditions of solvability were obtained and examples of boundary conditions were represented. Controllability conditions were obtained in the case when the corresponding set of controls are constant. We consider the so-called resonance ill-posed problem when the uniqueness can be disturbed, and corresponding linear interconnected system can have solutions not for any right-hand sides.

Keywords [1]
Moore-Penrose pseudoinverse operator, weakly nonlinear equation, boundary-value problem

1. Introduction

Weakly nonlinear boundary-value problems for the discrete equations plays an important role in the qualitative theory of dynamical systems. We consider conditions of the solvability for such interconnected systems with linear boundary conditions. Interconnected systems use as a model for investigations in applied sciences (see [1]). Moreover, the notion of input-to-state stability with corresponding estimates for such system is a very popular direction in the last years (see [2]). That's why we obtain input-to-state stability estimates and corresponding controllability conditions. It should be noted that this work is additional. We formulate general statement of the problem in nonlinear case but represent and prove the main results only in linear case. In the future works we use these results for investigating of nonlinear case and try to formulate chaotic conditions (in the weaklier sense than in [4]- [10]). It should be noted the papers [11], [12].

1.1. Statement of the problem

Consider the following interconnected system of nonlinear equations

$$x_i(n+1,\varepsilon) = A_i(n)x_i(n,\varepsilon) + B_i(n)u_i(n) + h_i(n) +$$
$$+ \varepsilon R_i\left(x_1(n,\varepsilon),\dots,x_{i-1}(n,\varepsilon),x_{i+1}(n,\varepsilon),\dots,x_p(n,\varepsilon)\right), n\epsilon Z, i=\overline{1,p} \quad (1)$$

with boundary conditions

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$$l_i x_i(\cdot,\varepsilon) = \alpha_i, \tag{2}$$

where $A_i(n), B_i(n) : B \to B$ - are a set of bounded operators, from the Banach space *B* into itself.

Assume that

$$A_i = \left(A_i(n)\right)_{n\in Z} \in l_\infty\left(Z, L(B)\right),\ B_i = \left(B_i(n)\right)_{n\in Z} \in l_\infty\left(Z, L(B)\right),$$

$$h_i = \left(h_i(n)\right)_{n\in Z} \in l_\infty(Z, B).$$

It means that:

$$\left|\left|A_i\right|\right| = sup_{n\in Z}\left|\left|A_i(n)\right|\right| < +\infty, \left|\left|h_i\right|\right| = sup_{n\in Z}\left|\left|h_i(n)\right|\right| < +\infty,$$

$$l_i: l_\infty(Z, B) \to BS$$

are the linear and bounded operators which translates bounded solutions of (1) into the Banach space *BS*, α_i are the elements of Banach space *BS*. We find conditions of the existence of bounded solutions (1), (2) which turn (ε = 0) in one of the bounded solutions of generating boundary-value problem

$$x_i^0(n+1) = A_i(n)x_i^0(n) + B_i(n)u_i(n) + h_i(n), n \in Z, i = \overline{1,p} \tag{3}$$

$$l_i x_i^0(\cdot) = \alpha_i. \tag{4}$$

First, we formulate the conditions for the existence of bounded solutions of the equation (3) and boundary value problem (3), (4). The corresponding homogeneous system of difference equations has the following form:

$$x_i(n+1) = A_i(n)x_i(n). \tag{5}$$

It should be noted that an arbitrary solution of a homogeneous system can be represented as: $x_i(m) = \Phi_i(m,n)x_i(n), m \geq n$, where:

$$\Phi_i(m,n) = \begin{cases} A_i(m-1)A_i(m-2)\ldots A_i(n), & if\ m > n \\ I, & if\ m = n \end{cases}.$$

It is clear, that

$$\Phi_i(m,0) = A_i(m-1)A_i(m-2)\ldots A_i(0).$$

Also, we denote

$$U_i(m) := \Phi_i(m,0) \text{ and } U_i(0) = I.$$

Traditionally [12], the mappings $\Phi_i(m,n)$ are called the **evolutionary operators** of the problem (5). Suppose that the system (5) is exponentially dichotomous [4, 12] on the semiaxes Z_+ and Z_- with projectors P_i and Q_i in the space *B* respectively, which means that there are projectors

$$P_i\ (P_i^2 = P_i) \text{ and } Q_i\ (Q_i^2 = Q_i),$$

constants

$$k_{1,2}^i \geq 1, 0 < \lambda_{1,2}^{(i)} < 1$$

such that

$$\begin{cases} \left|\left|U_i(n)P_iU_i^{-1}(m)\right|\right| \leq k_1^i\left(\lambda_1^{(i)}\right)^{n-m}, n \geq m \\ \left|\left|U_i(n)(I - P_i)U_i^{-1}(m)\right|\right| \leq k_1^i\left(\lambda_1^{(i)}\right)^{m-n}, m \geq n \end{cases},$$

for arbitrary $m, n \in Z_+$ (dichotomy on Z_+).

$$\begin{cases} \left|\left|U_i(n)Q_iU_i^{-1}(m)\right|\right| \leq k_2^i\left(\lambda_2^{(i)}\right)^{n-m}, n \geq m \\ \left|\left|U_i(n)(I - Q_i)U_i^{-1}(m)\right|\right| \leq k_2^i\left(\lambda_2^{(i)}\right)^{m-n}, m \geq n \end{cases},$$

for arbitrary $m, n \in Z_-$ (dichotomy on Z_-).

2. Main results. Linear case. Banach space case

In this part we obtain the necessary and sufficient conditions of the existence of the sets of bounded solutions for the linear generating boundary-value problems (3), (4) and controllability conditions.

2.1. Bounded solutions. Linear case

We formulate an auxiliary lemma which we will use when obtaining the main results (first lemma directly follows from the well-known results of [4], [12]).

Lemma 1. *Suppose that a homogeneous system is dichotomous on the semi-axes Z_+ and Z_- with projectors P_i and Q_i respectively, and the operators*

$$D_i = P_i - (I - Q_i): B \to B$$

are generalized invertible [5]. The solutions of the equation (1) bounded on the entire axis Z exist if and only if the following conditions are satisfied:

$$\sum_{k=-\infty}^{+\infty} H_i(k+1)\big(h_i(k) + B_i(k)u_i(k)\big) = 0. \qquad (6)$$

If the conditions (6) hold, the set of bounded solutions has the following view:

$$x_i^0 \ (n, c_i) = U_i(n)P_i P_{N(D_i)}c_i + (G_i[h_i + B_i u_i])(n), \quad c_i \in B \qquad (7)$$

where G_i are generalized inverse Green's operators [12, 13] on Z with the following properties:

$$(L_i G_i[h_i + B_i u_i])(n) = B_i(n)u_i(n) + h_i(n), n \in Z,$$

where

$(L_i x_i)(n) := x_i(n+1) - A_i(n)x_i(n): l_\infty(Z,B) \to l_\infty(Z,B)$,

$H_i(k+1) = P_{B_D} Q_i U_i^{-1}(k+1)$,

D_i^- are generalized inverse to the operators D_i,

projectors $P_{N(D_i)} = I - D_i^- D_i$ and $P_{B_{D_i}} = I - D_i D_i^-$ (see [3], [13]),

which project space B on the kernels $N(D_i)$ of the operators D_i and the subspaces $B_{D_i} := B \ominus R(D_i)$ respectively ($B = B_{D_i} \oplus R(D_i)$).

Remark 1. *We have the following estimates for the norm of the solutions:*

$$\left\|x_i^0(n,c_i)\right\| \le k_1^i(\lambda_1^i)^n \left\|P_{N(D_i)}c_i\right\| + k_1^i(\lambda_1^i)^n \|D_i^-\| \left(\frac{k_1^i\lambda_1^i}{1-\lambda_1^i} + \frac{k_2^i\lambda_2^i}{1-\lambda_2^i}\right)\left(\big\||h_i|\big\| + \big\||B_i|\big\|\,\big\||u_i|\big\|\right) +$$
$$k_1^i \frac{\left(1+\lambda_1^i-(\lambda_1^i)^n\right)}{1-\lambda_1^i}\left(\big\||h_i|\big\| + \big\||B_i|\big\|\,\big\||u_i|\big\|\right), n \ge 0, \qquad (8)$$

$$\left\|x_i^0(n,c_i)\right\| \le k_2^i(\lambda_2^i)^{-n} \left\|P_{N(D_i)}c_i\right\| + k_2^i(\lambda_2^i)^{-n} \|D_i^-\| \left(\frac{k_1^i\lambda_1^i}{1-\lambda_1^i} + \frac{k_2^i\lambda_2^i}{1-\lambda_2^i}\right)\left(\big\||h_i|\big\| +\right.$$
$$\left.\big\||B_i|\big\|\,\big\||u_i|\big\|\right) + k_2^i \frac{\left(1+\lambda_2^i-(\lambda_2^i)^{-n+1}\right)}{1-\lambda_2^i}\left(\big\||h_i|\big\| + \big\||B_i|\big\|\,\big\||u_i|\big\|\right), n \le 0. \qquad (9)$$

From these inequalities follows the estimates

$$\big\||x_i^0|\big\| \le K_i \left\|P_{N(D_i)}c_i\right\| + K_i\|D_i^-\| \left(\frac{k_1^i\lambda_1^i}{1-\lambda_1^i} + \frac{k_2^i\lambda_2^i}{1-\lambda_2^i}\right)\left(\big\||h_i|\big\| + \big\||B_i|\big\|\,\big\||u_i|\big\|\right) +$$
$$+ K_i \frac{1+\Lambda_1^i}{1-\Lambda_2^i}\left(\big\||h_i|\big\| + \big\||B_i|\big\|\,\big\||u_i|\big\|\right), \qquad (10)$$

where $K_i = max\{k_1^i, k_2^i\}$, $\Lambda_1^i = max\{\lambda_1^i, \lambda_2^i\}$, $\Lambda_2^i = min\{\lambda_1^i, \lambda_2^i\}$.

Remark 2. *It should also be noted that if bounded solutions are united together at zero as follows:*

$$x_i(0+) = x_i(0-) + c_i,$$

where c_i are the elements of Banach space, then we obtain a bounded solutions for the problem with a jump.

Let us find the solvability condition of the generating boundary-value problem (3), (4). Suppose that condition (6) is satisfied. Substitute expression (7) in the boundary condition (2). Then we obtain the operator equations which can be represented in the following form:

$$V_i c_i = \alpha_i - l_i(G_i[h_i + B_i u_i])(\cdot), \tag{11}$$

where $V_i = l_i U_i(\cdot) P_i P_{(N(D_i))}: B \to BS$. If the operators V_i are generalized invertible [6], then equation (11) is solvable [6] if and only if the following conditions are satisfied:

$$P_{V_{B_1}^i}\big(\alpha_i - l_i(G_i[h_i + B_i u_i])(\cdot)\big) = 0, \tag{12}$$

where $P_{V_{BS}^i} = I - V_i V_i^-$ $(BS = R(V_i) \oplus V_{B_{BS}^i})$. Under conditions (12) the sets of solutions of the system (11) have the following forms

$$c_i = V_i^-\big(\alpha_i - l_i(G_i[h_i + B_i u_i])(\cdot)\big) + P_{N(V_i)}\bar{c}_i, \qquad \bar{c}_i \epsilon B.$$

Thus, we obtain the following theorem.

Theorem 1. *Under condition (6) boundary-value problem (3), (4) has bounded solutions if and only if the conditions (12) are satisfied. The sets of bounded solutions have the following form:*

$$x_i(n, \bar{c}_i) = U_i(n) P_i P_{N(D_i)} P_{N(V_i)} \bar{c}_i + (\bar{G}_i[h_i + B_i u_i, \alpha_i])(n), \tag{13}$$

where $(\bar{G}_i[h_i + B_i u_i, \alpha_i])(n)$ are generalized Green's operators in the form

$$(\bar{G}_i[h_i + B_i u_i, \alpha_i])(n) = (G_i[h_i + B_i u_i])(n) +$$
$$+ U_i(n) P_i P_{N(D_i)} V_i^-\big(\alpha_i - l_i(G_i[h_i + B_i u_i])(\cdot)\big). \tag{14}$$

Remark 3. *It should be noted that the operators l_i in (2) can be for example represent two-point or multi-point boundary-value problems:*

$$l_i x_i(\cdot) = A_1^i x_i(m) - A_2^i x_i(0), \qquad l_i x_i(\cdot) = \sum_{j=1}^{n} C_i(j) x_i(m_j),$$

where $m, m_j \in Z$, $C_i(j) \in L(B, BS)$, $j = \overline{1,n}$ are linear and bounded operators. Another example is conditions at the infinity:

$$l_i x_i(\cdot) = A_1^i \lim_{n \to -\infty} x_i(n) + A_2^i \lim_{n \to +\infty} x_i(n).$$

Moreover, boundary-value problems (3), (4) in the represented form are the systems of independent boundary-value problems. We can consider more general linear boundary conditions (instead of (4)) in the forms:

$$\textstyle\sum_{k=1}^{p} l_k x_k^0(\cdot) = \rho, \;\; \rho \in BS. \tag{15}$$

In such way we obtain boundary-value problem with components of system x_i^0 which can connect by the condition (15).

2.2. Bounded solutions. Nonlinear case

Using theorem 1 we can obtain necessary condition of the existence of the sets of bounded solutions for the nonlinear boundary-value problem (1), (2). Suppose that generating boundary-value problem (3), (4) is solvable. It means that the conditions (6), (12) are satisfied. In this case we can easily obtain the following theorem.

Theorem 2. (Necessary condition). *Suppose that conditions (6), (12) (the sets of solutions (3), (4) has the form (13)) are satisfied and nonlinear problem (1), (2) has bounded solutions. Then $\bar{c}_i$ from the Banach space B (see (13)) satisfy the following operator system for the generating elements:*

$$F_i^1(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_p) := \sum_{k=-\infty}^{+\infty} H_i(k+1) R_i\big(x_1^0(k,\bar{c}_1), \dots, x_{i-1}^0(k,\bar{c}_{i-1}), x_{i+1}^0(k,\bar{c}_{i+1}), \dots, x_p^0(k,\bar{c}_p)\big) = 0$$

$$F_i^2(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_p) := P_{V_{B_1}^i}\,(l_i(G_i(R_i(x_1^0(\cdot,\bar{c}_1), \dots, x_{i-1}^0(\cdot,\bar{c}_{i-1}), x_{i+1}^0(\cdot,\bar{c}_{i+1}), \dots, x_p^0(\cdot,\bar{c}_p)\,))(\cdot)) = 0.$$

2.3. Controllability conditions

Consider the case when the controllability sequences $u_i(n) = u_i$ are fixed for any *n*. Then, conditions of reachability take the form of solvability of the following systems of operator equations:

$$Q_i u_i = g_i, \tag{16}$$

$$R_i u_i = w_i, \tag{17}$$

where the corresponding operators Q_i, R_i and elements g_i, w_i have the following form

$$Q_i = \sum_{k=-\infty}^{+\infty} H_i(k+1) B_i(k), \quad g_i = -\sum_{k=-\infty}^{+\infty} H_i(k+1) h_i(k),$$

$$R_i = P_{V_{B_1}^i} l_i(G_i[B_i])(\cdot), \qquad w_i = P_{V_{B_1}^i}\big(\alpha_i - l_i(G_i[h_i])(\cdot)\big).$$

We can rewrite the systems of operator equations (16), (17) in the form of matrix operator equation

$$u = G, \quad B_0 : B^p \to (B \times B_1)^p, \ u \in B^p, G \in (B \times B_1)^p, \tag{18}$$

where

$$B_0 = \begin{pmatrix} \begin{bmatrix} Q_1 \\ R_1 \end{bmatrix} & 0 & \dots & 0 \\ 0 & \begin{bmatrix} Q_2 \\ R_2 \end{bmatrix} & \dots & 0 \\ & \dots & & \\ 0 & 0 & \dots & \begin{bmatrix} Q_p \\ R_p \end{bmatrix} \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_p \end{pmatrix}, \qquad G = \begin{pmatrix} \begin{bmatrix} g_1 \\ w_1 \end{bmatrix} \\ \begin{bmatrix} g_2 \\ w_2 \end{bmatrix} \\ \dots \\ \begin{bmatrix} g_p \\ w_p \end{bmatrix} \end{pmatrix}.$$

If the operator B_0 is generalized invertible [6], then we can obtain the following theorem.

Theorem 2. (Reachability conditions). *Under condition*

$$P_{N(B_0^*)} G = 0, \tag{19}$$

the linear generating boundary-value problem (3), (4) is controllable. The sets of controls can be represented in the following form:

$$u = B_0^- G + P_{N(B_0)} v, \quad \textit{for any } v = (v_1^1, v_1^2, v_2^1, v_2^2, \dots, v_p^1, v_p^2) \in (B \times B_1)^p. \tag{20}$$

Remark 4. *Substituting solutions (20) in (13) we obtain the set of bounded solutions with corresponding set of controls u.*

Theorem 3. (Input-to-state stability conditions (see [1]-[3]). *It is easy to show that under conditions of the existence of bounded solutions from the inequalities (8), (9) we have the following estimates for any bounded solution of (3), (4)*

$$\left|\left|x_i^0(n,\bar{c}_i)\right|\right| \le M_i(\lambda_1^i)^n \left(\left|\left|P_{N(D_i)} P_{N(V_i)} \bar{c}_i\right|\right| + \left|\left|\alpha_i\right|\right|\right) + \gamma_i\left(\left|\left|h_i\right|\right|\right) + \gamma_i\left(\left|\left|B_i\right|\right| \left|\left|u_i\right|\right|\right), n \ge 0,$$

$$\left|\left|x_i^0(n,\bar{c}_i)\right|\right| \le M_i(\lambda_2^i)^{-n} \left(\left|\left|P_{N(D_i)} P_{N(V_i)} \bar{c}_i\right|\right| + \left|\left|\alpha_i\right|\right|\right) + \gamma_i\left(\left|\left|h_i\right|\right|\right) + \gamma_i\left(\left|\left|B_i\right|\right| \left|\left|u_i\right|\right|\right), n \le 0,$$

with corresponding constants M_i and functions γ_i.

Remark 5. *Presented theorem gives us conditions of the input-to-state stability and chaoticity (see also [1],[11]).*

Remark 6. *It should be noted that in more general case the linear and bounded operators l_i can translate bounded solutions of (1) into different spaces (instead of BS we can use BS_i, where BS_i are Banash spaces and $BS_i \neq BS_j$).*

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