On Existence of Global-in-Time Trajectories of Non-deterministic Markovian Systems

Ievgen Ivanov[1,2]

[1]Taras Shevchenko National University of Kyiv, Ukraine
[2]Paul Sabatier University, Toulouse, France
ivanov.eugen@gmail.com

Abstract. We consider the following question: given a continuous-time non-deterministic (not necessarily time-invariant) dynamical system, is it true that for each initial condition there exists a global-in-time trajectory. We study this question for a large class of systems, namely the class of complete non-deterministic Markovian systems. We show that for this class of systems, the question can be answered using analysis of existence of locally defined trajectories in a neighborhood of each time.

Keywords. dynamical systems, non-deterministic systems, Markovian systems, global-in-time trajectories

Key Terms. Mathematical Model, Specification Process, Verification Process

1 Introduction

In this paper we consider the following question: given a continuous-time non-deterministic (not necessarily time-invariant) dynamical system Σ, is it true that for any time moment t_0 and initial state x_0 there exists a global-in-time trajectory $t \mapsto s(t)$ such that $s(t_0) = x_0$.

Some related problems, e.g. global existence of solutions of initial value problems for various classes of differential equations [2, 3, 7] and inclusions [4–6], existence of non-Zeno global-in-time executions of hybrid automata [8–10] are well known. However, they have mostly been studied in the context of deterministic systems (differential equations with unique solutions, deterministic hybrid automata, etc.). Differential inclusions [5] are in principle non-deterministic systems, but for them a more common question is whether any (instead of some) solution for each initial condition exists into future [4, 6].

For deterministic systems the existence of a global trajectory for each initial condition implies that each partial trajectory (e.g. defined on a proper open interval of the real time scale) can be extended to a global trajectory. But this is not necessary for non-deterministic systems. For example, for each initial condition $x(t_0) = x_0$ the differential inclusion $\frac{dx}{dt} \in [0, x^2]$ has both a globally defined constant trajectory $x(t) = x_0$ and a trajectory of the equation $\frac{dx}{dt} = x^2$ which escapes to infinity in finite time. Thus it is not true that any (locally defined) solution extends infinitely into future.

We will study our existence question for a large class of systems, namely the class of complete non-deterministic Markovian systems. We will show that for this class of systems, the question can be answered using analysis of existence of locally defined trajectories in a neighborhood of each time.

Note that in this paper we use the term Markovian in the context of purely non-deterministic (i.e. non-stochastic) systems. The formal definition will be given below. Also note that many well-known classes of continuous-time systems either belong to this class of can be represented by systems of this class. We will give examples later in the paper.

2 Non-deterministic Complete Markovian Systems

The notions of a Markov process or system [12] are usually defined and studied in the context of probability theory. However, they also make sense in a purely non-deterministic setting, where no quantitative information is attached to events (trajectories, transitions, etc.), i.e. each event is either possible or impossible.

General definitions of continuous-time Markovian systems of such kind have appeared in the literature [1]. They give a large class of (not necessarily deterministic) systems which can have both continuous and discontinuous (jump-like) trajectories. Essentially, the notion of a non-deterministic Markovian system captures the idea that only the system's current state (but not its past) determines the set of its possible futures.

Below we define the notion of a non-deterministic (complete) Markovian system in spirit of, but not exactly as in [1]. The main reasons for this are that we would like to include non-time-invariant systems in the definition and focus on partial trajectories, i.e. trajectories defined on a subset of the time scale.

We will use the following notation: $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $f : A \to B$ is a total function from A to B, $f : A \tilde{\to} B$ is a partial function from A to B, $f|_X$ is the restriction of a function f to a set X, 2^A is the power set of a set A. The notation $f(x) \downarrow$ ($f(x) \uparrow$) means that $f(x)$ is defined (resp. undefined) on the argument x, $dom(f) = \{x \mid f(x) \downarrow\}$. Also, $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ denote the logical operations of negation, disjunction, conjunction, implication and equivalence correspondingly. Let us denote:

- $T = [0, +\infty)$ is the (real) time scale. We assume that T is equipped with a topology induced by the standard topology on $\mathbb{R}$
- $\mathfrak{T}$ is the set of all connected subsets of T with cardinality greater than one.

For the purpose of this paper, we will use the following definition of a dynamical system on the time scale T.

Definition 1. *A dynamical system on T is as an abstract object M (a mathematical model; in applications this may be an equation, inclusion, switched system, etc.) together with the associated time scale T (this scale will be the same* *throughout the paper), the set of states Q, and the set of (partial) trajectories Tr. A trajectory is a function $s : A \to Q$, where $A \in \mathfrak{T}$ (note that trivial trajectories defined on singleton or empty time sets are excluded). The set Tr satisfies the property: if $s : A \to Q \in Tr$, $B \in \mathfrak{T}$, and $B \subseteq A$, then $s|_B \in Tr$. We will refer to this property as "Tr is closed under proper restrictions (CPR)".*

We will say that a trajectory $s_1 \in Tr$ is a *subtrajectory* of $s_2 \in Tr$ (denoted as $s_1 \sqsubseteq s_2$), if $s_1 = s_2|_A$ for some $A \in \mathfrak{T}$. The trajectories s_1 and s_2 are *incomparable*, if s_1 is not a subtrajectory of s_2 and vice versa.

According to the definition given above, for a time $t_0 \in T$ and $q_0 \in Q$ there may exist multiple incomparable trajectories s such that $s(t_0) = q_0$ (as well as one or none). In this sense a dynamical system can be non-deterministic.

It is easy to see that $(Tr, \sqsubseteq)$ is a partially ordered set (poset).

Definition 2. *A set Tr (which is CPR) is*

- *complete, if $(Tr, \sqsubseteq)$ is a chain-complete poset (every chain has a supremum)*
- *Markovian, if $s \in Tr$ for each $s_1, s_2 \in Tr$ and $t \in T$ such that $t = \sup dom(s_1) = \inf dom(s_2)$, $s_1(t)\downarrow$, $s_2(t)\downarrow$, and $s_1(t) = s_2(t)$, where* $s(t) = \begin{cases} s_1(t), & t \in dom(A) \\ s_2(t), & t \in dom(B) \end{cases}$.

Note that because Tr is closed under restrictions to sets $A \in \mathfrak{T}$, the supremum of a chain c in poset $(Tr, \sqsubseteq)$ exists iff $s_* \in Tr$, where $s_* : \bigcup_{s \in c} dom(s) \to Q$ is defined as follows: $s_*(t) = s(t)$, if $s \in c$ and $t \in dom(s)$ (this definition is correct, because c is a chain with respect to subtrajectory relation).

The notions of complete and Markovian sets of trajectories are illustrated in Fig. 1 and 2.

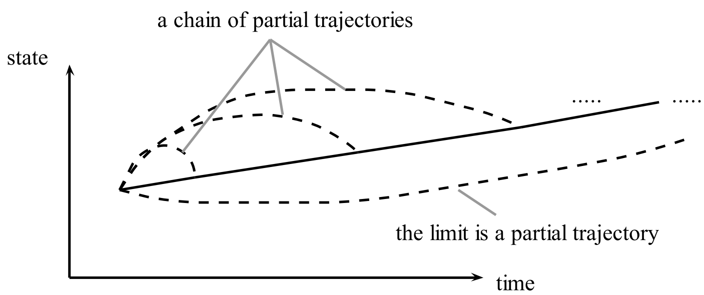

Fig. 1. Completeness property

The following proposition gives some examples of sets of trajectories.

Proposition 1. *Let $Q = \mathbb{R}$. Consider the following sets of trajectories:*

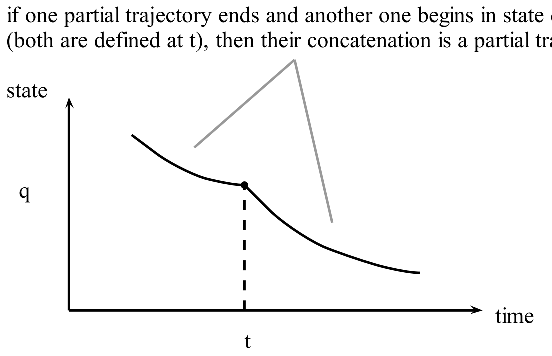

Fig. 2. Markovian property

- Tr_{all} *is the set of all functions* $s : A \to Q$, $A \in \mathfrak{T}$.
- Tr_{cont} *is the set of all continuous functions* $s \in Tr_{all}$ *(on their domains)*
- Tr_{diff} *is the set of all differentiable functions* $s \in Tr_{all}$ *(on their domains)*
- Tr_{bnd} *is the set of all bounded functions* $s \in Tr_{all}$ *(on their domains).*

Then the following holds:

(1) $\emptyset$, Tr_{all}, Tr_{cont}, Tr_{diff}, Tr_{bnd}, $Tr_{diff} \cap Tr_{bnd}$ *are CPR*
(2) $\emptyset$, Tr_{all}, Tr_{cont} *are complete and Markovian*
(3) Tr_{diff} *is complete, but is not Markovian*
(4) Tr_{bnd} *is Markovian, but is not complete*
(5) $Tr_{diff} \cap Tr_{bnd}$ *is neither complete, nor Markovian.*

Definition 3. *A non-deterministic complete Markovian system (NCMS) is dynamical system is* (M, T, Q, Tr) *such that* Tr *is complete and Markovian.*

The following propositions 2-4 give some examples of NCMS.

Proposition 2. *Let* $Q = \mathbb{R}^d$ $(d \in \mathbb{N})$ *and* M *be a differential equation* $\frac{dy}{dt} = f(t, y)$, *where* $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *is a given total function. Let* Tr *be the set of all functions* $s : A \to Q$, $A \in \mathfrak{T}$ *such that* s *is differentiable on* A *and satisfies* M *on* A. *Then* (M, T, Q, Tr) *is a NCMS.*

Proposition 3. *Let* M *be a differential inclusion* $\frac{dy}{dt} = F(t, y)$, *where* $F : \mathbb{R} \times \mathbb{R}^d \to 2^{\mathbb{R}^d}$ *is a given (total) function. This is not necessarily a NCMS, but it can be converted to a NCMS as follows. Let* M' *be the system* $\begin{cases} \frac{dy}{dt} = x \\ y \in F(t, x) \end{cases}$, *where* x *is a new variable. Let* $Q = \mathbb{R}^d \times \mathbb{R}^d$ *and* Tr *be the set of all* $s : A \to Q$, $A \in \mathfrak{T}$ *such that* s *is locally absolutely continuous on* A *and satisfies* M' *almost everywhere on* A *(w.r.t. Lebesgue's measure). Then* (M, T, Q, Tr) *is a NCMS.*

Proposition 4. *Let Q be a set equipped with discrete topology. Let $r \subseteq Q \times Q$ be a relation on Q. Let M be a system* $\begin{cases} y(t+) = y(t), & t \notin \mathbb{N}_0 \\ (y(t), y(t+)) \in r, & t \in \mathbb{N}_0 \end{cases}$, *where y denotes an unknown function, $y(t+)$ denotes the right limit at t. Let Tr be the set of all piecewise-constant left-continuous functions $s : A \to Q$ (w.r.t. discrete topology on Q) which satisfy M on A (see Fig. 3). Then (M, T, Q, Tr) is a NCMS.*

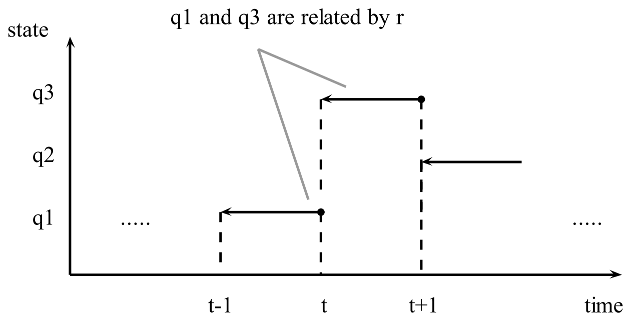

Fig. 3. A piecewise-constant left-continuous trajectory which models an execution of a (discrete-time) transition system (Q, r).

Below we will describe a general complete Markovian set of trajectories (or a system) in terms of certain local predicates.

Let us introduce the following terminology:

Definition 4. *Let $s_1, s_2 : T \tilde{\to} Q$. Then s_1 and s_2:*

- *coincide on a set $A \subseteq T$, if $A \subseteq dom(s_1) \cap dom(s_2)$ and $s_1(t) = s_2(t)$ for each $t \in A$. We denote this as $s_1 \doteq_A s_2$.*
- *coincide in a left neighborhood of $t \in T$, if $t > 0$ and there exists $t' \in [0, t)$, such that $s_1 \doteq_{(t',t]} s_2$. We denote this as $s_1 \doteq_{t-} s_2$.*
- *coincide in a right neighborhood of $t \in T$, if there exists $t' > t$, such that $s_1 \doteq_{[t,t')} s_2$. We denote this as $s_1 \doteq_{t+} s_2$.*

Let Q be a set of states. Denote by $ST(Q)$ the set of pairs (s, t) where $s : A \to Q$ for some $A \in \mathfrak{T}$ and $t \in A$.

Definition 5. *A predicate $p : ST(Q) \to Bool$ ($Bool = \{true, false\}$) is called*

- *left-local, if $p(s_1, t) \Leftrightarrow p(s_2, t)$ whenever $(s_1, t), (s_2, t) \in ST(Q)$ and $s_1 \doteq_{t-} s_2$, and moreover, $p(s, t)$ whenever t is the least element of $dom(s)$*
- *right-local, if $p(s_1, t) \Leftrightarrow p(s_2, t)$ whenever $(s_1, t), (s_2, t) \in ST(Q)$, $s_1 \doteq_{t+} s_2$, and moreover, $p(s, t)$ whenever t is the greatest element of $dom(s)$*

– *left-stable, if whenever t is not the least element of $dom(s)$, $p(s,t)$ implies that there exists $t' \in [0,t)$ such that $p(s,\tau)$ for all $t' \in [t',t] \cap dom(s)$*
– *right-stable, if whenever t is not the greatest element of $dom(s)$, $p(s,t)$ implies that there exists $t' > t$ such that $p(s,\tau)$ for all $\tau \in [t,t'] \cap dom(s)$.*

The theorems given below show how left- and right-local predicates can be used to specify/represent a complete Markovian set of trajectories (or system).

Theorem 1. *Let $l : ST(Q) \to Bool$ be a left-local predicate and $r : ST(Q) \to Bool$ be a right-local predicate. Then the set*

$$Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l(s,t) \wedge r(s,t))\}.$$

is CPR, complete, and Markovian.

Theorem 2. *Let Tr be a CPR complete Markovian set of trajectories which take values in the set of states Q. Then there exist unique predicates $l, r : ST(Q) \to Bool$ such that l is left-local and left-sable, r is right-local and right-stable, and*

$$Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l(s,t) \wedge r(s,t))\}.$$

Let us consider an example which illustrates these theorems. Let $Q = \mathbb{R}^d$ and Tr be the set of all functions $s : A \to Q$, $A \in \mathfrak{T}$ such that s is differentiable on A and satisfies a differential equation $\frac{dy}{dt} = f(t,y)$ on A, where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a given function. Then Tr is complete and Markovian by Proposition 2.

Let us show how Tr can be represented using left- and right-local predicates. Let $l, r : ST(Q) \to Bool$ be predicates such that

– $l(s,t)$ iff either t is the least element of $dom(s)$, or $\partial_- s(t) \downarrow = f(t, s(t))$,
– $r(s,t)$ iff either t is the greatest element of $dom(s)$, or $\partial_+ s(t) \downarrow = f(t, s(t))$,

where $\partial_- s(t)$ and $\partial_+ s(t)$ denote the left- and right- derivative of s at t (the symbol $\downarrow$ indicates that the left hand side of the equality is defined). It is not difficult to check that l is left-local, r is right-local, and $Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l(s,t) \wedge r(s,t))\}$. In general case, l and r are not necessarily (respectively) left- and right-stable. But we can define another predicates l_*, r_* on $ST(Q)$ such that

– $l_*(s,t)$ iff either t is the least element of $dom(s)$, or there exists $t' < t$ such that s is differentiable on $(t',t]$ and satisfies differential equation $\frac{dy}{dt} = f(t,y)$ on $(t',t]$ (at the time t the derivative is understood as left-derivative).
– $r_*(s,t)$ iff either t is the greatest element of $dom(s)$, or there exists $t' > t$ such that s is differentiable on $[t,t')$ and satisfies $\frac{dy}{dt} = f(t,y)$ on $[t,t')$.

Then it is not difficult to see that l_* is left-local and left-stable, and r_* is right-local and right-stable, and $Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l(s,t) \wedge r(s,t))\}$.

3 Existence of Global-in-Time Trajectories

Let us recall our original question about global-in-time trajectories and formulate it for non-deterministic complete Markovian systems.

Let (M, T, Q, Tr) be a NCMS. Our question (let us denote it as **Q0**) is whether it is true that for each $t_0 \in T$, $q_0 \in Q$ there exists a trajectory $s : T \to Q$ (i.e. global-in-time) such that $s(t_0) = q_0$.

Note that we ask about existence of a trajectory defined in both time directions relative to t_0. The case when we are interested in existence of a trajectory defined in one direction (e.g $s : [t_0, +\infty) \to Q$) is not considered in this paper, but can be studied analogously.

Let us decompose **Q0** into the following two questions:

Q1: Is it true that for each $t_0 \in T$, $q_0 \in Q$ there exists a (partial) trajectory $s : A \to Q$ such that t_0 is an interior point of A (relative to the topology on T, e.g. 0 is considered an interior point of $[0, 1]$) and $s(t_0) = q_0$?

Q2: Is it true that for each partial trajectory $s : A \to Q$ such that A is a compact segment there exists a trajectory $s' : T \to Q$ such that $s = s'|_A$?

Proposition 5. *The answer to the question **Q0** is positive iff the answers to **Q1** and **Q2** are positive.*

The question **Q1** is about existence of a local-in-time trajectories. We will not study it in this paper and assume that it can be answered using domain-specific methods. Our aim is to answer **Q2** using only information about existence of locally defined trajectories in the neighborhood of each time moment.

Let us introduce several definitions. Let $\Sigma = (M, T, Q, Tr)$ be a fixed NCMS.

Definition 6.
- *A right dead-end path (in Σ) is a trajectory $s : A \to Q$ such that A has a form $[a, b)$, where $a, b \in T$. and there is no $s' : [a, b] \to Q \in Tr$ such that $s = s'|_{dom(s)}$ (i.e. s cannot be extended to a trajectory on $[a, b]$). The value b is called the end of this path.*
- *A left dead-end path (in Σ) is a trajectory $s : A \to Q$ such that A has a form $(a, b]$, where $a, b \in T$. and there is no $s' : [a, b] \to Q \in Tr$ such that $s = s'|_{dom(s)}$. The value a is called the end of this path.*
- *A dead-end path is either a right dead-end path, or a left dead-end path.*

Let $f : [0, +\infty) \to [0, +\infty)$ be a positive-definite (i.e. $f(0) = 0$, $f(x) > 0$ when $x > 0$), monotonously non-decreasing, and continuous function (e.g. $f(x) = x$).

Definition 7.
- *A right dead-end path $s : [a, b) \to Q$ is called f-O_b-escapable, where O_b is a connected neighborhood of b, if there exists $c \in (a, b) \cap O_b$, $d \in [b + f(b - c), +\infty)$, and a trajectory $s' : [c, d] \cap O_b \to Q$ such that $s(c) = s'(c)$.*
- *A left dead-end path $s : (a, b] \to Q$ is called f-O_b-escapable, where O_b is a connected neighborhood of b, if there exists $c \in (a, b) \cap O_b$, $d \in [0, \max\{a - f(c - a), 0\}]$, and a trajectory $s' : [d, c] \cap O_b \to Q$ such that $s(c) = s'(c)$.*
- *A right- or left- dead-end path is called f-escapable, if it is f-T-escapable.*

This definition is illustrated in Fig. 4. Note that a suffix of a right dead-end path $s : [a, b) \to Q$ (i.e. a restriction of the form $s|_{[a',b)}$, where $a' \in [a, b)$) is a right dead-end path. Analogously, a prefix of a left dead-end path $s : (a, b] \to Q$ (i.e. a restriction of the form $s|_{(a,b']}$, where $b' \in (a, b]$) is a left dead-end path.

Let $\Sigma = (M, T, Q, Tr)$ be a NCMS. For each $t \in T$ let $O_t \subseteq T$ be some connected neighborhood of t and D_t be the set of all dead-end paths s (in Σ) such that t is the end of s and $dom(s) \subseteq O_t$.

Theorem 3. *The following conditions are equivalent:*

(1) for each partial trajectory $s : A \to Q$ such that A is a compact segment there exists a trajectory $s' : T \to Q$ such that $s = s'|_A$
(2) each dead-end path (in Σ) is f-escapable
(3) for each $t \in T$ and $s \in D_t$, s is f-O_t-escapable.

Note that this theorem holds for an arbitrary fixed f and arbitrary fixed choice of neighborhoods O_t, $t \in T$.

This theorem gives an answer to the question **Q2**. The condition 3 of this theorem shows in which sense Theorem 3 reduces the question of global-in-time existence of trajectories to the analysis of local existence of trajectories in the neighborhood of each time moment.

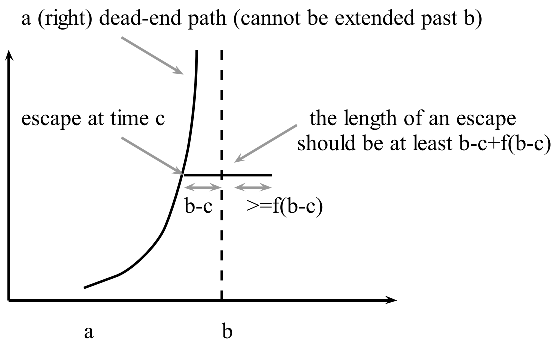

Fig. 4. An f-escapable right dead-end path.

4 Conclusion

We have studied the question of existence of global-in-time trajectories for each initial condition of a (non-time-invariant) non-deterministic complete Markovian system. We have shown that this question can be answered using analysis of existence of locally defined trajectories in a neighborhood of each time. The results can be useful for studying the problems of well-posedness and reachability for continuous and discrete-continuous (hybrid) dynamical systems.

References

1. Willems, J.: Paradigms and Puzzles in the Theory of Dynamical Systems. IEEE Transactions on automatic control. 36, 259–294 (1991)
2. Coddington, E., Levinson, N.: Theory of ordinary differential equations. McGraw-Hill, New York (1955)
3. Fillipov, A.: Differential equations with discontinuous right-hand sides. AMS Trans. 42, 199–231 (1964)
4. Tangiguchi, T.: Global existence of solutions of differential inclusions. Journal of mathematical analysis and applications. 166, 41–51 (1992)
5. Aubin, A., Cellina, A.: Differential inclusions. Springer-Verlag, Berlin (1984)
6. Seah, S.W.: Existence of solutions and asymptotic equilibrium of multivalued differential systems. J. Math. Anal. Appl. 89, 648–663 (1982)
7. Gliklikh, Y.: Necessary and sufficient conditions for global-in-time existence of solutions of ordinary, stochastic, and parabolic differential equations. Abstract and Applied Analysis. 2006, 1–17 (2006)
8. Heemels, W., Camlibel, M., Van der Schaft, A.J., Schumacher, J.M.: On the existence and uniqueness of solution trajectories to hybrid dynamical systems. In: Johansson, R., Rantzer, A. (eds.) Nonlinear and Hybrid Control in Automotive Applications. pp. 391–422. Springer, Berlin (2003)
9. Goebel, R., Sanfelice, R., Teel, R.: Hybrid dynamical systems. IEEE Control Systems Magazine. 29, 29–93 (2009)
10. Henzinger, T.: The theory of hybrid automata. IEEE Symposium on Logic in Computer Science. 278–292 (1996)
11. Constantin, A.: Global existence of solutions for perturbed differential equations. Annali di Matematica Pura ed Applicata. 168, 237–299 (1995)
12. Doob, J.B.: Stochastic processes. Wiley-Interscience (1990)