The epistemic structure of de Finetti's betting problem

Tommaso Flaminio[1] and Hykel Hosni[2]

[1] IIIA - CSIC
Campus de la Univ. Autònoma de Barcelona s/n 08193 Bellaterra, Spain.
Email: tommaso@iiia.csic.es
[2] Scuola Normale Superiore,
Piazza dei Cavalieri 7 Pisa, Italy.
Email hykel.hosni@sns.it

Abstract. De Finetti's conception of events is one of the most distinctive aspects of his theory of probability, yet it appears to be somewhat elusive. The purpose of this note is to set up a formal framework in which a rigorous characterisation of this notion, and its cognate modelling assumptions, gives rise to a detailed formalisation of the betting problem which underlies the celebrated Dutch Book Argument. As our main result shows, this refinement captures an intuitive condition which de Finetti imposed on the betting problem, namely that it is irrational to bet on an event which may be true, but whose truth will never be ascertained by the players.

Keywords: Uncertain reasoning, Events; De Finetti's betting problem; Partial valuations; Epistemic structures.

1 Introduction and motivation

Let $E_1, \ldots, E_n$ be of events of interest. De Finetti's *betting problem* is the choice that an idealised agent called *bookmaker* must make when publishing a *book*, i.e. when making the assignment $B = \{(E_i, \beta_i) : i = 1, \ldots, n\}$ such that each E_i is given value $\beta_i \in [0, 1]$. Once a book has been published, a *gambler* can place bets on event E_i by paying $\alpha_i \beta_i$ to the bookmaker. In return for this payment, the gambler will receive α_i, if E_i occurs and nothing otherwise. De Finetti constructs the betting problem in such a way as to force the bookmaker to publish *fair betting odds* for book B. To this end, two modelling assumptions are built into the problem, namely that (i) the bookmaker is forced to accept any number of bets on B and (ii) when betting on E_i, gamblers can choose the sign, as well as the magnitude of (monetary) stakes α_i. Conditions (i-ii) force the bookmaker to publish books with zero-expectation, for doing otherwise may offer the gambler the possibility of making a sure profit, possibily by choosing negative stakes thereby unilaterally imposing a payoff swap to the bookmaker. As the game is zero-sum, this is equivalent to forcing the bookmaker into sure loss. In this context, de Finetti proves that the axioms of probability are necessary and sufficient to secure the bookmaker against this possibility.

The crux of the Dutch book argument is the identification of the agent's degrees of belief with the *price* they are willing to pay for an uncertain reward which depends on the *future* truth value of some *presently unknown* propositions – the *events* on which the agents are betting. This clearly suggests that the semantics of events, which bears directly on the definition of probability, is implicitly endowed with an *epistemic structure*. The purpose of this paper is to give this structure an explicit formal characterisation and to show how the resulting framework helps us making de Finetti's elusive notion of *event* much clearer. In particular we shall be able to give a formal setting to the following remark:

> [T]he characteristic feature of what I refer to as an "event" is that the circumstances under which the event will turn out to be "verified" or "disproved" have been fixed in advance. [1]

1.1 Formal preliminaries

Let $L = \{q_1, \ldots, q_n\}$ be a classical propositional language. The set of sentences $SL = \{\varphi, \psi, \theta, \ldots\}$ is inductively built up from L through the propositional connectives $\wedge, \vee, \rightarrow$, and $\neg$, as usual. We use $\perp$ to denote *falsum*. *Valuations* are mappings ω form L into $\{0, 1\}$ that naturally extend to SL by the truth-functionality of the propositional connectives (with the usual stipulation that $\omega(\perp) = 0$, for all valuations ω). We denote by $\Omega(L)$ the class of all valuations over L.

A *partial valuation* on L is a map $\nu : X \subseteq L \rightarrow \{0, 1\}$ which, for $\omega \in \Omega$ and for every $q \in L$ is defined by

$$\nu(q) = \begin{cases} \omega(q) & \text{if } q \in X; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The class of all partial valuations over L is denoted by $\Omega^P(L)$. For $\nu : X \rightarrow \{0, 1\}$ and $\mu : Y \rightarrow \{0, 1\}$ in $\Omega^P(L)$ we say that μ *extends* ν (and we write $\nu \subseteq \mu$) if $X \subseteq Y$, and for every $x \in X$, $\nu(x) = \mu(x)$.

Finally, for every formula φ we set $[\varphi] = \{\psi \in SL : \vdash \varphi \leftrightarrow \psi\}$, where as usual, $\vdash$ denotes the classical provability relation. We conform to the custom of referring to the equivalence classes $[\varphi]$ as to *the proposition* φ.

Note that de Finetti's notion of event is captured by propositions, rather than sentences. In fact, in the present logical framework, the "circumstances" under which the propositions turn out to be true or false are nothing but the valuations in $\Omega(L)$. As a consequence, de Finetti's notion of *book* is formally defined by (probability) assignments on a (finite) set of propositions. We will reserve the expression *propositional books* to refer to these particular assignments.

Definition 1. *Let $\varphi \in SL$, and let $\nu \in \Omega^P(L)$. (1) We say that ν* realizes *φ, written $\nu \Vdash \varphi$, if $\nu(\varphi)$ is defined.*

(2) We say that ν realizes *$[\varphi]$, written $\nu \Vdash [\varphi]$, if there exists at least a $\psi \in [\varphi]$ such that $\nu \Vdash \psi$. In this case we assign $\nu(\gamma) = \nu(\psi)$ for every $\gamma \in [\varphi]$.*

Let W be a finite set of nodes interpreted, as usual, as possible worlds. Let $e : W \to \Omega^P(L)$ such that for every $w \in W$, $e(w) = \nu_w : X_w \subseteq L \to \{0,1\}$ is a partial valuation. Note that to avoid cumbersome notation we will write ν_w instead of $e(w)$ to denote the partial valuation associated to w, and similarly, we denote by X_w the subset of L for which ν_w is defined. Finally, let $R \subseteq W \times W$ be an *accessibility relation*. We call a triplet (W, e, R) a *partially evaluated frame* (**pef** for short).

Let $w \in W$ and let $[\varphi]$ be a proposition. We say that that w *decides* $[\varphi]$, if $\nu_w \Vdash [\varphi]$.

Definition 2 (Events and Facts). *Let (W, e, R) be a* **pef***, $w \in W$ and $\varphi \in SL$. Then we say that a proposition $[\varphi]$ is:*

- *A w-*event *iff $\nu_w \nVdash [\varphi]$, and for every $\nu \in \Omega^P(L)$ such that $\nu \supseteq \nu_w$ and $\nu \Vdash [\varphi]$, there exists w' such that $R(w, w')$, and $\nu_{w'} = \nu$.*
- *A w-*fact *iff $\nu_w \Vdash [\varphi]$.*

For every $w \in W$ we denote by $\mathcal{E}(w)$ and $\mathcal{F}(w)$ the classes of w-events, and w-facts respectively.

An important consequence of Definition 2 is that in **pef** events (and facts) are relativised to a specific state of the world.

A **pef** $K = (W, e, R)$ is said to be *monotonic* if satisfies:

(M) for every $w, w' \in W$, if $R(w, w')$, then $\nu_w \subseteq \nu_{w'}$.

Recall that in de Finetti's intuitive characterisation, at the time (i.e. world in W) at which the contract is signed, bookmaker and gambler agree on which conditions will realise the events in the book. This clearly presupposes some form of monotonic persistence of the underlying structure, which Property **(M)** guarantees. In particular, in every monotonic **pef**, w-facts are w'-facts in each w' which is accessible from w. In addition their truth value once determined, is fixed throughout the frame.

Definition 3. *Let (W, e, R) be any monotonic* **pef***, and let $w \in W$. A w-book is a propositional book $B = \{[\varphi_i] = \beta_i : i = 1, \ldots, n\}$ where the propositions $[\varphi_i]$ are w-events.*

Finally, we say that a **pef** (W, e, R) is *complete* if the following property is satisfied:

(C) for every $\varphi \in SL$ and for every $\nu \in \Omega^P$ such that $\nu \Vdash [\varphi]$, there exists a $w_{[\varphi]} \in W$ such that $\nu = \nu_{w_{[\varphi]}} \Vdash [\varphi]$, i.e. $[\varphi]$ is a $w_{[\varphi]}$-fact.

Definition 4 (Inaccessible propositions). *Let (W, e, R) be a* **pef** *and $w \in W$. A proposition $[\varphi]$ is said to be w-*inaccessible *if $\nu_w \nVdash [\varphi]$ and for every w' such that $\nu_{w'} \Vdash [\varphi]$, $\neg R(w, w')$.*

Inaccessible propositions relativise the betting problem to the specific information available to the gambler and the bookmaker. This is clearly in consonance with de Finetti's rather strict subjectivism according to which all that matters for the determination of the final payoff is that the agents *agree* on which events are realised.

2 No bets on inaccessible propositions

Our first result singles out the conditions under which a coherent w-book B can be extended, either by w-facts or by w-inaccessible propositions to a coherent propositional book B'.

Theorem 1. *Let $B = \{[\varphi_i] = \beta_i : i = 1, \ldots, n\}$ be a coherent w-book, let $[\psi_1], \ldots, [\psi_r]$ be propositions that are not w-events, and let $B' = B \cup \{[\psi_j] = \gamma_j : j = 1, \ldots, r\}$ be a propositional book extending B. Then the following hold:*

(1) If all propositions $[\psi_j]$ are w-facts, then B' is coherent if and only if for every $j = 1, \ldots, r, \gamma_j = \nu_w(\psi_j)$;
(2) If all propositions $[\psi_j]$ are w-inaccessible, then B' is coherent if and only if for every $j = 1 \ldots, r$, $\gamma_j = 0$.

Proof. We only prove the direction from left-to-right, the converse being is immediate in both cases.

(1) Suppose, to the contrary, that exists j such that, $\gamma_j \neq \nu_w(\psi_j)$, and in particular suppose that $\nu_w(\psi_j) = 1$, so that $\gamma_j < 1$. Then, the gambler can secure sure win by betting a positive α on ψ_j. In this case in fact, since the **pef** is monotonic by the definition of w-book, $\nu_{w'}(\varphi_i) = 1$ holds in every world w' which is accessible from w. Thus the gambler pays $\alpha \cdot \gamma_j$ in order to surely receive α in any w' accessible from w. Conversely, if $\nu_w(\psi_j) = 0$, then $\gamma_j > 0$ and in that case it is easy to see that a sure-winning choice for the gambler consists in swapping payoffs with the bookmaker, i.e. to bet a negative amount of money on $[\psi_j]$.

(2) As above suppose to the contrary that $\gamma_j > 0$ for some j, and that the gambler bets $-\alpha$ on $[\psi_j]$. By contract, this means that the bookmaker must pay $\alpha \cdot \gamma_j$ to the gambler, thus incurring sure loss, since $[\psi_j]$ will not be decided in any world w' such that $R(w, w')$. □

Theorem 1 captures the key property identified by de Finetti in his informal characterisation of events, namely that *no monetary betting is rational* unless the conditions under which the relevant events will be decided are known to the bookmaker and the gambler.

3 The language of w-events

A gambler and a bookmaker interpreted on a complete and monotonic **pef** are guaranteed that: (1) as soon as a proposition is realized in w, it will stay so across the accessible worlds from w (monotonicity), and (2) for every sentence φ, there exists a world w that realizes $[\varphi]$ (completeness). In accordance with the above informal discussion of the betting problem, not only gamblers and bookmaker must agree that a world w in which the events of interest are realized exists. They also must agree on the conditions under which this will happen, as captured by Theorem 1.

Example 1 ([2]). Consider an electron ϵ, and a world w. We are interested the position and the energy of ϵ at w. Let $[\varphi]$ and $[\psi]$ be the propositions expressing those measurements, respectively. Moreover let us assume that both $[\varphi]$ and $[\psi]$ are w-events. Indeed if at w we are uncertain about the position and the energy of ϵ, we can certainly perform experiments to determine them. But, what about $[\varphi] \wedge [\psi]$? Position and energy are represented by non-commuting operators in quantum theory, and we can assign an electron a definite position and a definite energy, but not both. This fact can be modelled in complete and monotonic **pef** K, by forcing $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$ to be w-inaccessible.

Definition 5. *A* **pef** (W, e, R) *is* fully accessible *if R satisfies:*

(**A**) *for all $w, w' \in W$, if $\nu_w \subseteq \nu_{w'}$, then $R(w, w')$.*

It is customary [3] to characterise the coherence of a propositional book B in terms of its extension to a (finitely additive) probability measure on the Boolean algebra spanned by the events in B. As our second result shows, completeness and full accessibility are sufficient conditions in order for the algebra generated by the w-events in a w-book B to contain *only* w-events, and therefore, situations like that of Example 1 cannot be modelled in this context.

Theorem 2. *Let $K = (W, e, R)$ be a complete* **pef**. *If K is fully accessible, then for every $w \in W$, $\mathcal{E}(w)$ is closed under the classical connectives.*

Proof. Let w be any world such that $\mathcal{E}(w) \neq \emptyset$ and let $[\varphi_1]$ and $[\varphi_2]$ be w-events. Without loss of generality we can assume that φ_1 and φ_2 are written in conjunctive normal form. We want to prove that $[\varphi_1] \wedge [\varphi_2] = [\varphi_1 \wedge \varphi_2]$ is a w-event. Clearly $\nu_w \not\Vdash [\varphi_1 \wedge \varphi_2]$, and hence we only need to show that for every partial valuation $\nu' \in \Omega^P$ such that $\nu' \supseteq \nu_w$, and $\nu' \Vdash [\varphi_1 \wedge \varphi_2]$, there exists a world w^* such that $R(w, w^*)$, and $\nu' = \nu_{w^*}$.

For every $\nu' \in \Omega^P$ such that $\nu' \Vdash [\varphi_1 \wedge \varphi_2]$, (**C**) ensures the existence of a w^* (that we would have denoted $w_{[\varphi_1 \wedge \varphi_2]}$ using the terminonlogy of (**C**)) such that $\nu^* \Vdash [\varphi_1 \wedge \varphi_2]$, and $\nu_w \subseteq \nu_{w^*} = \nu'$. Since (W, e, R) satisfies (**A**), $\nu_w \subseteq \nu_{w^*}$ ensures $R(w, w^*)$ and hence our claim is settled.

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