A Tool-Based Set Theoretic Framework for Concept Approximation

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Abstract. Modelling positive and negative knowledge has a long-standing tradition in Formal Concept Analysis. To approximate concepts we propose a tool-based set theoretic partial approximation framework in which positive features and their negative counterparts of observed objects can be approximated simultaneously.

Key words: Concept approximation, positive-negative knowledge, rough set theory, partial approximation framework

1 Introduction

Modelling of learning from positive and negative examples has a long-standing tradition in machine learning, for a brief historical survey see, e.g. [16]. A possible model in terms of Formal Concept Analysis was described in [10, 15].

The idea of knowing negatively was introduced explicitly by M. Minsky in [19]. Negative knowledge has a number of beneficial effects in professional contexts which are discussed in detailed, e.g. in [12, 19]. The adjectives 'positive' and 'negative' do not imply a valuation *per se*. 'Positive' knowledge is not good, advantageous or benign, whereas 'negative' knowledge is not bad, disadvantageous or malign in and of itself.

Both positive and negative knowledge have procedural [19] and declarative aspects [23]. A procedural aspect of positive and negative knowledge can be paraphrased as 'to know what to do' and 'to know what not to do' resp., whereas a declarative aspect as 'to know what one knows' and 'to know what one does not know' resp. In addition, both positive and negative knowledge have two different degrees of knowing or not-knowing. Positive knowledge is informed (uninformed) when one is (not) aware of his/her own relevant knowledge. Negative knowledge is informed (uninformed) when one is (not) aware of his/her own lack of relevant knowledge. Our discussion deals with the declarative aspect of positive and negative knowledge and informed way of knowing/not-knowing.

In our approach, first, we consider a class of objects which is modelled as an abstract set, called the universe of discourse. We assume that a concept is defined over the universe as a subset. In real life the concepts are usually expressed in natural language and so their exact definition cannot be given. The concept approximation is a fundamental problem in artificial intelligence in order to be able to solve real-world problems [17, 22, 31]. A possible way to approximate concepts is to induce their approximations from available experimental (observed, measured) data which is also modelled as subsets over the universe [29–31]. Concepts are generally rough, whereas measurements are always crisp.

It is also assumed that we have some well-defined, decidable features with which an observed object possesses or not. These features assign crisp subsets within the universe. In other words, we model an object of interest as a member of an abstract set, called the universe, and its property 'it possesses a feature' as 'it is the element of a crisp subset of the universe'.

In practice, a concept, of course, cannot be specified completely over the universe. Instead, two relevant sample groups of objects can be established determined by our currently available and necessarily constrained knowledge: a group of which members characteristically possess some features concerning the concept in question, and another group of which members do not substantially possess the same features. Both groups correspond two crisp subsets of the universe. They are disjoint, and, in general, the union of them does not add up to the whole universe. For obvious reasons, the former can be marked with the adjective positive and called the *positive sample set*, whereas the latter with negative and called the *negative sample set*.

Moreover, in real life, a feature of objects cannot be observed directly as well. We need *tools* at our disposal with which we are able to measure one or more constituents of a feature which are called *properties*. For instance, let us say that we observe velocity (feature) of cars (objects). Velocity is a vector quantity with *speed* and *direction*. They are two properties of velocity which can be measured simultaneously and both of them can be expressed numerically. And so, a car is modelled as a member of an abstract set, the universe, and its velocity as it is a member of intersection of two subsets of cars with given speed and given direction (tools) which were measured at the same time.

It is assumed that we are able to judge easily and unambiguously whether an object possesses a property ascertained by a tool or not. It is expected that tools can be used simply and quickly. The objects classified by a tool can be modelled as a crisp subset of the universe. With a slight abuse of terminology, this subset is also simply called tool.

Different tools form different subsets, but they are not necessarily disjoint. Intersections of not disjoint tools are also viewed as tools. The complement of a tool is not necessarily a tool at the same time. In practice, there are properties which can be measured but their counterparts cannot. For instance, a given disease can be diagnosed but the health cannot be measured. These significant facts confirm the partial nature of our approach.

Let us distinguish two types of tools: *positive* and *negative* ones. It is a natural assumption that a subset cannot be positive and negative tool simultaneously.

To manage the problem outlined above we need an *approximation framework.* It may be built on the *rough set theory* because it provides a powerful foundation to reveal and discover important structures in data and classify complex objects [27, 28]. The rough set theory was introduced by the Polish mathematician, Z. Pawlak in the early 1980s [24, 25]. It can be seen as a new mathematical approach to *vagueness* [14]. According to Pawlak's idea, the vagueness of a subset within the universe U is defined by the difference of its upper and lower approximations with respect to a partition of U. Using partitions, however, is a very strict requirement. Our starting point is an *arbitrary* family of subsets of U which does not cover the universe necessarily. The lower and upper approximations are straightforward point-free generalizations of Pawlak's ones [2–6]. We apply them to build a *set theoretic tool-based partial approximation framework* in which positive features and their negative counterparts of any clump of observed objects can be approximated *simultaneously.*

The rest of the paper is organized as follows. Section 2 sums up the most important features of rough set theory and partial approximation spaces. Classical rough set theory and formal concept analysis use similar structures to represent information which is briefly described in Section 3. In Section 4 we will propose a tool-based set theoretical framework for concept approximation based on partial approximation spaces. Its main notions are illustrated in Section 5. Finally, in Section 6, we conclude the paper.

2 Partial Approximation of Sets

First, we summarize the most important concepts and properties of rough set theory [13, 25]. Let U be a nonempty set and ε be an equivalence relation on U. Let U/ε denote the partition of U generated by ε. Members of the partition are called *ε-elementary sets.* $X \subseteq U$ is *ε-definable*, if it is a union of ε-elementary sets, otherwise *ε-undefinable.* By definition, the empty set is considered to be an ε-definable set.

The pair $\langle U, \varepsilon \rangle$ is called a Pawlakean approximation space. The lower and upper ε-approximations of $X \subseteq U$ can be defined as follows.

The *lower ε-approximation* of X is[3]

$$\underline{\varepsilon}(X) = \bigcup\{Y \mid Y \in U/\varepsilon, Y \subseteq X\},$$

and the *upper ε-approximation* of X is

$$\overline{\varepsilon}(X) = \bigcup\{Y \mid Y \in U/\varepsilon, Y \cap X \neq \emptyset\}.$$

The set $B_\varepsilon(X) = \overline{\varepsilon}(X) \setminus \underline{\varepsilon}(X)$ is the *ε-boundary* of X. X is *ε-crisp*, if $B_\varepsilon(X) = \emptyset$, otherwise X is *ε-rough.*

[3] If $\mathfrak{A} \subseteq 2^U$, we define $\bigcup \mathfrak{A} = \{x \mid \exists A \in \mathfrak{A}(x \in A)\}$, and $\bigcap \mathfrak{A} = \{x \mid \forall A \in \mathfrak{A}(x \in A)\}$. If $\mathfrak{A}$ is an empty family of sets, $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = U$.

Let $\mathfrak{D}_{U/\varepsilon}$ denote the family of ε-definable subsets of U. Clearly, $\underline{\varepsilon}(X), \overline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$, and the maps $\underline{\varepsilon}, \overline{\varepsilon} : 2^U \to \mathfrak{D}_{U/\varepsilon}$ are monotone, total and many-to-one. It can easily be seen ([25], Proposition 2.2, points 1, 9, 10) that the map $\underline{\varepsilon}$ is *contractive* and $\overline{\varepsilon}$ is *extensive*, i.e. $\forall X \in 2^U (\underline{\varepsilon}(X) \subseteq X \subseteq \overline{\varepsilon}(X))$. In other words, X is bounded by its lower and upper approximations.

Now, let us turn to the theory of partial approximation of sets [2, 4, 5]. Its most fundamental notion is the base system.

Definition 1. *Let $\mathfrak{B} \subseteq 2^U$ be a nonempty family of nonempty subsets of U.*
$\mathfrak{B}$ *is called the* base system*, its members are the* $\mathfrak{B}$-sets*.*

An extension of the base system is specified by the next definition.

Definition 2. *A nonempty subset $X \subseteq U$ is* $\mathfrak{B}$-definable *if there exists a family of sets $\mathfrak{D} \subseteq \mathfrak{B}$ in such a way that $X = \bigcup \mathfrak{D}$, otherwise X is* $\mathfrak{B}$-undefinable*.*
The empty set is considered to be a $\mathfrak{B}$-definable set.
Let $\mathfrak{D}_{\mathfrak{B}}$ denote the family of $\mathfrak{B}$-definable sets of U.

Note that neither the base system $\mathfrak{B}$ nor $\mathfrak{D}_{\mathfrak{B}}$ covers the universe necessarily.

Let us define the lower and upper approximations of sets based on partial covering of the universe.

Definition 3. *Let $\mathfrak{B} \subseteq 2^U$ be a base system and X be any subset of U.*
The lower $\mathfrak{B}$-approximation of X *(Fig. 1) is*

$$\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \{Y \mid Y \in \mathfrak{B}, Y \cap X = Y\},$$

the upper $\mathfrak{B}$-approximation of X *(Fig. 2) is*

$$\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \bigcup \{Y \mid Y \in \mathfrak{B}, Y \cap X \neq \emptyset\}.$$

Remark 1. In Definition 3, the members of the base system may be seen as the elements of the lattice 2^U, and instead of set theoretic operations may be used lattice operations. In this way, point-free generalizations of Pawlakean lower and upper approximations can be obtained.

Clearly, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X), \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) \in \mathfrak{D}_{\mathfrak{B}}$, and the maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} : 2^U \to \mathfrak{D}_{\mathfrak{B}}$ are total, monotone and in general many-to-one.

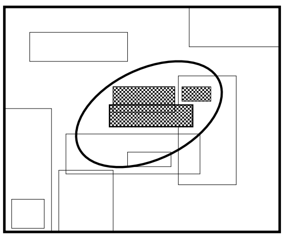


Fig. 1. Lower approximation

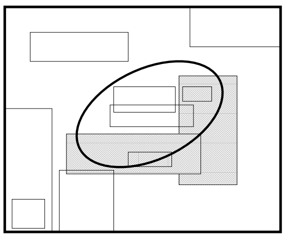


Fig. 2. Upper approximation

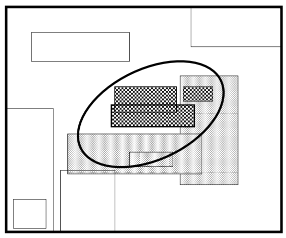


Fig. 3. Lower and upper approximations

Proposition 1 ([6], Proposition 4.8). *Let $\mathfrak{B} \subseteq 2^U$ be a base system. Then*

1. $\forall X \in 2^U(\mathfrak{C}^\flat_{\mathfrak{B}}(X) \subseteq \mathfrak{C}^\sharp_{\mathfrak{B}}(X))$*;*
2. $\forall X \in 2^U(\mathfrak{C}^\flat_{\mathfrak{B}}(X) \subseteq X)$*—that is, $\mathfrak{C}^\flat_{\mathfrak{B}}$ is contractive;*
3. $\forall X \in 2^U(X \subseteq \mathfrak{C}^\sharp_{\mathfrak{B}}(X))$ *if and only if $\bigcup \mathfrak{B} = U$—that is, $\mathfrak{C}^\sharp_{\mathfrak{B}}$ is extensive if and only if $\mathfrak{B}$ covers the universe.*

Using the previous notations, the notion of the partial approximation space can be introduced.

Definition 4. *The ordered quadruple $\langle U, \mathfrak{D}_{\mathfrak{B}}, \mathfrak{C}^\flat_{\mathfrak{B}}, \mathfrak{C}^\sharp_{\mathfrak{B}}\rangle$ is called the* (weak) partial $\mathfrak{B}$-approximation space.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets.

Definition 5. *The pair of maps $f : P \to Q$ and $g : Q \to P$ forms a* (regular) Galois connection *between P and Q, in notation $\mathbb{G}(P, f, g, Q)$, if*

$$\forall p \in P\, \forall q \in Q\ (f(p) \leq_Q q \Leftrightarrow p \leq_P g(q)).$$

If $P = Q$, $\mathbb{G}(P, f, g, P)$ is called a Galois connection on P.

Remark 2. Here we adopted the definition of Galois connection in which the maps are monotone. It is also called monotone or covariant form. For more details on Galois connections, see, e.g. [8]. Note that since Galois connections are not necessarily symmetric, the order of the maps is important.

It is well known fact ([13], Proposition 138) that upper and lower ε-approximations form a Galois connection $\mathbb{G}(2^U, \overline{\varepsilon}, \underline{\varepsilon}, 2^U)$ on $(2^U, \subseteq)$. Next theorem shows the conditions under which upper and lower $\mathfrak{B}$-approximations also form a Galois connection.

Theorem 1 ([6], Theorem 4.14). *Let $\langle U, \mathfrak{B}, \mathfrak{C}^\flat_{\mathfrak{B}}, \mathfrak{C}^\sharp_{\mathfrak{B}}\rangle$ be a partial $\mathfrak{B}$-approximation space. The upper and lower $\mathfrak{B}$-approximations form a Galois connection $\mathbb{G}(2^U, \mathfrak{C}^\sharp_{\mathfrak{B}}, \mathfrak{C}^\flat_{\mathfrak{B}}, 2^U)$ on $(2^U, \subseteq)$ if and only if the base system $\mathfrak{B}$ is a partition of U.*

According to Proposition 1, point 3, $X \subseteq \mathfrak{C}^\sharp_{\mathfrak{B}}(X)$ if and only if the base system $\mathfrak{B}$ covers the universe.

Definition 6. *A subset $X \subseteq U$ is* $\mathfrak{B}$-approximatable *if $X \subseteq \mathfrak{C}^\sharp_{\mathfrak{B}}(X)$, otherwise it is said that X has a* $\mathfrak{B}$-approximation gap.

A $\mathfrak{B}$-approximation gap may be interpreted so that our knowledge about the universe encoded in the base system is incomplete and not enough to approximate X.

Definition 7. *Let $\langle 2^U, \mathfrak{D}_{\mathfrak{B}}, \mathfrak{C}^\flat_{\mathfrak{B}}, \mathfrak{C}^\sharp_{\mathfrak{B}}\rangle$ be a partial $\mathfrak{B}$-approximation space, and X be any subset of U.*

The partial upper $\mathfrak{B}$-approximation *of X is*

$$\partial\mathfrak{C}^\sharp_{\mathfrak{B}}(X) = \begin{cases} \mathfrak{C}^\sharp_{\mathfrak{B}}(X), & \text{if } X \text{ is } \mathfrak{B}\text{-approximatable;} \\ \text{undefined, otherwise.} \end{cases} \tag{1}$$

There exists at least one nonempty $B \in \mathfrak{B}$ $\mathfrak{B}$-set by Definition 2. Then $B \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(B)$ according to Definition 3. Hence, $\partial\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ is defined on at least one nonempty subset of U.

Notice that $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) \subseteq X \subseteq \partial\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)$ holds provided X is $\mathfrak{B}$-approximatable.

As Theorem 1 shows, the upper and lower $\mathfrak{B}$-approximations form a Galois connection on $(2^U, \subseteq)$ if and only if the base system $\mathfrak{B}$ is a partition of U. The question naturally arises whether the Galois connection could be generalized so that the maps $\partial\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ and $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ may form a Galois connection in any sense. Moreover, if the answer is yes, then what conditions have to be fulfilled by a partial $\mathfrak{B}$-approximation space so that $\partial\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ and $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ form a Galois connection of this special type. Recall that $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ is a total and $\partial\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ is a partial map on 2^U.

To answer this question, first of all, we need a suitable modified notion of Galois connections.

Definition 8 ([18], Definition 2.2.2). *The pair of maps $f : P \to Q$ and $g : Q \to P$ forms a* partial Galois connection *between P and Q, denoted by $\partial\mathbb{G}(P, \partial f, g, Q)$, if*

1. *$f : P \to Q$ is a monotone partial map,*
2. *$g : Q \to P$ is a monotone total map,*
3. *$f(g(q))$ exists for all $q \in Q$, and*
4. *$\forall p \in P$ and $\forall q \in Q$ such that $f(p)$ is defined, $f(p) \leq_Q q \Leftrightarrow p \leq_P g(q)$.*

Remark 3. In [18], A. Miné actually introduced the concept of $\mathcal{F}$-partial Galois connection $\partial\mathbb{G}(P, \partial f, g, Q)$ between the concrete domain P and the abstract domain Q, where $\mathcal{F}$ is a set of concrete operators. We apply this notion in the simplest form when $P = Q = 2^U$ and $\mathcal{F} = \emptyset$. It is allowed by Miné's definition.

Theorem 2 ([6], Theorem 4.22). *Let $\langle U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}}\rangle$ be a partial $\mathfrak{B}$-approximation space.*

The partial upper $\mathfrak{B}$-approximation and the lower $\mathfrak{B}$-approximation form a partial Galois connection $\partial\mathbb{G}(2^U, \partial\mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, 2^U)$ on $(2^U, \subseteq)$ if and only if the $\mathfrak{B}$-sets are pairwise disjoint.

A natural question is how we can form a base system from an arbitrary one of which members are pairwise disjoint. In practice, this problem can be reduced to finite base systems. A possible way to construct such a base system is the following.

First, let us form an intersection structure from an arbitrary finite base system. Formally, a nonempty family $\mathfrak{S} \subseteq 2^U$ is an intersection structure if $\forall \mathfrak{S}'(\neq \emptyset) \subseteq \mathfrak{S}\,(\bigcap \mathfrak{S}' \in \mathfrak{S})$, i.e. it is closed under intersection but does not contain U necessarily [7].

Let us take an arbitrary finite base system $\mathfrak{B}$ and create its intersection structure $IS(\mathfrak{B})$ as the smallest set which satisfies the following two properties:

1. $\mathfrak{B} \subseteq IS(\mathfrak{B})$.
2. If $\mathfrak{B}' \subseteq IS(\mathfrak{B})$, then $\bigcap \mathfrak{B}' \in IS(\mathfrak{B})$.

Having given the intersection structure $IS(\mathfrak{B})$, we can create a family of sets $IS_{\Pi}(\mathfrak{B})$ of which members are pairwise disjoint. $IS_{\Pi}(\mathfrak{B})$ is the smallest family of sets which satisfies the following property:

$$\text{If } u \in U \text{ and } \mathfrak{B}' = \{B : B \in \mathfrak{B} \wedge u \in B\}, \text{ then } \bigcap \mathfrak{B}' \in IS_{\Pi}(\mathfrak{B}).$$

3 Rough Set Theory and Formal Concept Analysis

Let G and M denote a set of objects and a finite set of attributes, respectively. Note that the formal concept analysis allows that G and M to be empty sets, but the rough set theory does not.

3.1 Information Systems

First, we reformulate the rough set theory [9, 24]. Let $S = \langle G, M, V_{m \in M}, f \rangle$ be an *information system*, where G and M as before, V_m is a nonempty set of values of attribute $m \in M$, and $f : G \times M \to V = \bigcup_{m \in M} V_m$ is an information function with $\forall g \in G \forall m \in M\, (f(g, m) \in V_m)$. Informally, $f(g, m)$ represents the value which object g takes at attribute m.

The information system is often represented by a table, as shown in Table 1. It is an information table containing a shortened student grade history from an information technology course held for hospital nurses at the Faculty of Health, University of Debrecen. It contains 20 students and their results in three homework assignments, and one final examination.

Table 1. Information system of a shortened student grade history (complete)

Student	Hw1	Hw2	Hw3	**Final exam**
S_1	1	1	1	**1**
S_2	1	1	2	**2**
S_3	1	1	1	**1**
S_4	1	2	1	**1**
S_5	1	1	1	**1**
S_6	1	1	2	**1**
S_7	4	1	3	**1**
S_8	2	4	1	**2**
S_9	1	3	1	**2**
S_{10}	1	1	3	**1**
S_{11}	2	1	1	**2**
S_{12}	1	1	1	**1**
S_{13}	1	2	1	**1**
S_{14}	1	1	2	**3**
S_{15}	4	3	3	**4**
S_{16}	2	1	1	**4**
S_{17}	2	2	2	**4**
S_{18}	4	4	3	**3**
S_{19}	4	3	3	**2**
S_{20}	4	4	3	**4**

Table 2. Information system of a shortened student grade history (partial)

Student	Hw1	Hw2	Hw3	**Final exam**
S_1	1	1	1	**1**
S_2	1	1	2	**2**
S_3	1	1	1	**1**
S_4		2		**1**
S_5	1	1	1	**1**
S_6	1	1	2	**1**
S_7		1	3	**1**
S_8	2	4	1	**2**
S_9	1	3	1	**2**
S_{10}	1	1	3	**1**
S_{11}	2	1	1	**2**
S_{12}	1	1	1	**1**
S_{13}	1	2	1	**1**
S_{14}	1	1	2	**3**
S_{15}	4	3	3	**4**
S_{16}	2	1	1	**4**
S_{17}				**4**
S_{18}	4	4	3	**3**
S_{19}	4	3	3	**2**
S_{20}	4	4	3	**4**

With each $N \subseteq M$ we associate an equivalence relation $E_N \subseteq G \times G$ by

$$(g_1, g_2) \in E_N \text{ if } \forall n \in N\,(f(g_1, n) = f(g_2, n)).$$

If $g \in G$, then $[g]_{E_N}$ is the equivalence class of E_N containing g. Let G/N denote the set of equivalence classes generated by E_N.

A concept $X \subseteq G$ is E_N-definable or E_N-exact if X is a union of some equivalence classes, otherwise X is E_N-undefinable or E_N-inexact.

Lower and upper E_N-approximations of X are:

$$\underline{E_N}(X) = \bigcup\{[g]_{E_N} \in G/N \mid [g]_{E_N} \subseteq X\},$$
$$\overline{E_N}(X) = \bigcup\{[g]_{E_N} \in G/N \mid [g]_{E_N} \cap X \neq \emptyset\}.$$

3.2 Formal Context

In formal concept analysis a *formal context* is a triple $\langle G, M, R\rangle$ [11], where G and M as above and $R \subseteq G \times M$ is a binary relation. Choosing

$$\forall m \in M\,(V_m = \{0, 1\}) \text{ and } f(g, m) = \begin{cases} 1, \text{ if } (g, m) \in R; \\ 0, \text{ otherwise} \end{cases},$$

we may transform information systems into formal contexts. For instance, Table 3 shows a formal context representation of the same example shown in Table 1.

Table 3. Formal context of a shortened student grade history

	Homework1					Homework2					Homework3					Final
Student	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	exam
S_1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_2	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	**2**
S_3	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_4	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	**1**
S_5	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_6	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	**1**
S_7	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	**1**
S_8	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	**2**
S_9	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	**2**
S_{10}	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	**1**
S_{11}	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	**2**
S_{12}	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_{13}	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	**1**
S_{14}	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	**3**
S_{15}	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	**4**
S_{16}	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	**4**
S_{17}	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	**4**
S_{18}	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	**3**
S_{19}	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	**2**
S_{20}	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	**4**

Table 4. Incomplete formal context of a shortened student grade history

	Homework1					Homework2					Homework3					Final
Student	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	exam
S_1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_2	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	**2**
S_3						1	0	0	0	0						**1**
S_4	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	**1**
S_5	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_6	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	**1**
S_7						1	0	0	0	0	0	0	1	0	0	**1**
S_8	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	**2**
S_9	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	**2**
S_{10}	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	**1**
S_{11}	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	**2**
S_{12}	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	**1**
S_{13}	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	**1**
S_{14}	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	**3**
S_{15}	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	**4**
S_{16}	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	**4**
S_{17}																**4**
S_{18}	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	**3**
S_{19}	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	**2**
S_{20}	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	**4**

Given the formal context $\langle G, M, R\rangle$ we define

$$A^{\rhd} = \{m \in M \mid \forall g \in A\,((g,m) \in R)\}, \text{ for } A \subseteq G,$$
$$B^{\lhd} = \{g \in G \mid \forall m \in B((g,m) \in R)\}, \text{ for } B \subseteq M,$$

called the *polars* of A, B, respectively [26].

Informally, $A^{\rhd}$ is the set of attributes common to all the objects in A, $B^{\lhd}$ is the set of all objects which possess all of the attributes in B.

Given $A \subseteq G$ and $B \subseteq M$ we have $A \times B \subseteq R \Leftrightarrow A \subseteq B^{\lhd} \Leftrightarrow A^{\rhd} \supseteq B$. The pair (A, B) is called a *formal concept* if

$$A = B^{\lhd} \text{ and } A^{\rhd} = B.$$

Formal concepts are usually ordered by inclusion on the first co-ordinate and/or reverse inclusion on the second:

$$(A_1, B_1) \preceq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \text{ and } B_1 \supseteq B_2 \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2.$$

Formal concepts with this ordering form a concept hierarchy for the context $\langle G, M, R\rangle$ and denoted by $\mathcal{B}(G, M, R)$. The fundamental theorem of formal concept analysis states that $\mathcal{B}(G, M, R)$ with the ordering $\preceq$ is a complete lattice called the *concept lattice* [11].

4 A Tool-Based Set Theoretic Approximation Framework

Let U be any nonempty set. Let $A^+, A^- \subseteq U$ be two nonempty subsets of U in such a way that $A^+ \cap A^- = \emptyset$. A^+ and A^- are called the *positive* and *negative reference set*, respectively. The adjectives *positive* and *negative* claim nothing else but that the sets A^+ and A^- are well separated.

In general, the constraint $A^+ \cap A^- = \emptyset$ is the only requirement for A^+ and A^-. Of course, additional relations between them may be supposed.

Furthermore, let $\mathfrak{T}^+$ and $\mathfrak{T}^- \subseteq 2^U$ be two nonempty *finite* families of subsets of U. The members of $\mathfrak{T}^+$ are called *positive* or $\mathfrak{T}^+$*-tools*, whereas the members of $\mathfrak{T}^-$ are called *negative* or $\mathfrak{T}^-$*-tools*.

Requirements for positive and negatives tools are the following:

(T1) For each subset $T^+ \in \mathfrak{T}^+$ (resp., $T^- \in \mathfrak{T}^-$) it is *easy* to decide whether an element of U belongs to T^+ (resp., T^-) or not.
(T2) Sets in $\mathfrak{T}^+$ are not necessarily pairwise disjoint, neither are those in $\mathfrak{T}^-$.
(T3) $\mathfrak{T}^+ \cap \mathfrak{T}^- = \emptyset$.
(T4) Neither $\bigcup \mathfrak{T}^+$ nor $\bigcup \mathfrak{T}^-$ covers U necessarily.
(T5) It is assumed that

$$\forall T_1^+, T_2^+ \in \mathfrak{T}^+\,(T_1^+ \cap T_2^+ \in \mathfrak{T}^+), \text{ and } \forall T_1^-, T_2^- \in \mathfrak{T}^-\,(T_1^- \cap T_2^- \in \mathfrak{T}^-),$$

i.e. the $\mathfrak{T}^+$ and $\mathfrak{T}^-$ are closed under intersection.

Positive (resp., negative) tools provide an opportunity to locate or approximate the positive (resp., negative) reference set. Positive and negative tools together also yield useful information about the reference sets. To do this, we can use the following three partial approximation spaces relaying on $\mathfrak{T}^+$ and $\mathfrak{T}^-$:

$$\langle U, \mathfrak{D}_{\mathfrak{T}^+}, \mathfrak{C}^\flat_{\mathfrak{T}^+}, \mathfrak{C}^\sharp_{\mathfrak{T}^+}\rangle,\ \langle U, \mathfrak{D}_{\mathfrak{T}^-}, \mathfrak{C}^\flat_{\mathfrak{T}^-}, \mathfrak{C}^\sharp_{\mathfrak{T}^-}\rangle,\ \langle U, \mathfrak{D}_{\mathfrak{T}^+\cup\mathfrak{T}^-}, \mathfrak{C}^\flat_{\mathfrak{T}^+\cup\mathfrak{T}^-}, \mathfrak{C}^\sharp_{\mathfrak{T}^+\cup\mathfrak{T}^-}\rangle.$$

Within these spaces, any clump of observed objects can be approximated with the help of the lower and upper $\mathfrak{T}^+$($\mathfrak{T}^-$-,$\mathfrak{T}^+ \cup \mathfrak{T}^-$-)-approximations.

5 An Illustrative Example

To illustrate our framework let us see a simple example. We want to approximately estimate the achievement of students and their results in the final examination in higher education [20, 21]. We have at our disposal an information table (Table 5) containing the student grade history (5 = excellent, 4 = good, 3 = fair, 2 = pass, 1 = fail).

Table 5. Information table with student grade history

Student	Hw1	Hw2	Hw3	**Final exam**
S_1	1	1	1	**1**
S_2	1	1	2	**2**
S_3	1	1	1	**1**
S_4	1	2	1	**1**
S_5	1	1	1	**1**
S_6	1	1	2	**1**
S_7	4	1	3	**1**
S_8	2	4	1	**2**
S_9	1	3	1	**2**
S_{10}	1	1	3	**1**
S_{11}	2	1	1	**2**
S_{12}	1	1	1	**1**
S_{13}	1	2	1	**1**
S_{14}	1	1	2	**3**
S_{15}	4	3	3	**4**
S_{16}	2	1	1	**4**
S_{17}	2	2	2	**4**
S_{18}	4	4	3	**3**
S_{19}	4	3	3	**2**
S_{20}	4	4	3	**4**

Positive tools:

$T^+_{Hw1=4} = \{S_7, S_{15}, S_{18}, S_{19}, S_{20}\}$

$T^+_{Hw2=4} = \{S_8, S_{18}, S_{20}\}$

$T^+_{Hw1=4 \wedge Hw2=4} = \{S_{18}, S_{20}\}$

Negative tools:

$T^-_{Hw1=1} =$
$\{S_1, S_2, S_3, S_4, S_5, S_6, S_9, S_{10}, S_{12}, S_{13}, S_{14}\}$

$T^-_{Hw2=1} =$
$\{S_1, S_2, S_3, S_5, S_6, S_7, S_{10}, S_{11}, S_{12}, S_{14}, S_{16}\}$

$T^-_{Hw3=1} =$
$\{S_1, S_3, S_4, S_5, S_8, S_9, S_{11}, S_{12}, S_{13}, S_{16}\}$

$T^-_{Hw1=1 \wedge Hw2=1} =$
$\{S_1, S_2, S_3, S_5, S_6, S_{10}, S_{12}, S_{14}\}$

$T^-_{Hw1=1 \wedge Hw3=1} = \{S_1, S_3, S_4, S_5, S_9, S_{12}, S_{13}\}$

$T^-_{Hw1=2 \wedge Hw3=1} = \{S_1, S_3, S_5, S_{11}, S_{12}, S_{16}\}$

$T^-_{Hw1=1 \wedge Hw2=1 \wedge Hw3=1} = \{S_1, S_3, S_5, S_{12}\}$

Of course, there is no way to accurately measure the achievement of students and their success or failure on the final exam. Moreover, students cannot exactly appreciate 'what they know' or 'what they do not know'. However, with the apparatus of partial approximation spaces, we can analyze student grade history contained in Table 5 in order to understand how the results in assignments approximately relate to success or failure on the final exam.

For the sake of simplicity, students' success and failure on homework assignments or the final exam are measured by grade 4 and grade 1, respectively. Based on these prerequisites, the positive tools (Fig. 4) and negative tools (Fig. 5) are the following (see also Table 5):

$$\begin{aligned}\mathfrak{T}^+ &= \{T^+_{Hw1=4},\, T^+_{Hw2=4},\, T^+_{Hw1=4\wedge Hw2=4}\},\\ \mathfrak{T}^- &= \{T^-_{Hw1=1},\, T^-_{Hw2=1}, T^-_{Hw3=1},\, T^-_{Hw1=1\wedge Hw2=1}, T^-_{Hw1=1\wedge Hw3=1},\\ &\quad T^-_{Hw1=2\wedge Hw3=1},\, T^-_{Hw1=1\wedge Hw2=1\wedge Hw3=1}\}\end{aligned}$$

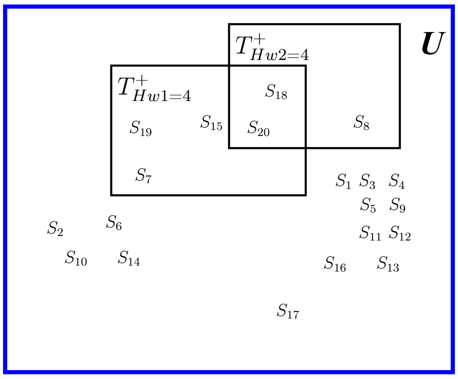


Fig. 4. Positive tools (homework = 4). $\langle U, \mathfrak{D}_{\mathfrak{T}^+}, \mathfrak{C}^\flat_{\mathfrak{T}^+}, \mathfrak{C}^\sharp_{\mathfrak{T}^+}\rangle$ partial approximation space

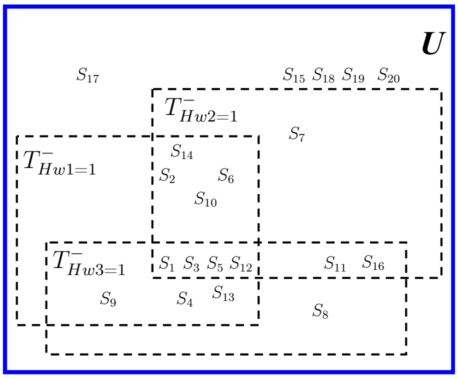


Fig. 5. Negative tools (homework = 1). $\langle U, \mathfrak{D}_{\mathfrak{T}^-}, \mathfrak{C}^\flat_{\mathfrak{T}^-}, \mathfrak{C}^\sharp_{\mathfrak{T}^-}\rangle$ partial approximation space

Students who have successful final exams can be evaluated with both positive and negative tools (Fig. 6, Fig. 7):

- $\mathfrak{C}^\flat_{\mathfrak{T}^+}(X_{Final\_exam=4}) = \emptyset$
  Informally: there is no combination of successful homework in which case the final exam *surely* succeeds.
- $\mathfrak{C}^\sharp_{\mathfrak{T}^+}(X_{Final\_exam=4}) = T^+_{Hw1=4} \cup T^+_{Hw2=4} \cup T^+_{Hw1=4\wedge Hw2=4}$
  Informally: if one of the Homework 1, 2 or both of the two succeed, the final exam *possibly* succeeds.
- $\mathfrak{C}^\flat_{\mathfrak{T}^-}(X_{Final\_exam=4}) = \emptyset$
  Informally: there is no combination of failed homework in which case the final exam *surely* succeeds.
- $\mathfrak{C}^\sharp_{\mathfrak{T}^-}(X_{Final\_exam=4}) = T^-_{Hw3=1}$
  Informally: if the Homework 3 fails, the final exam *may* succeed.

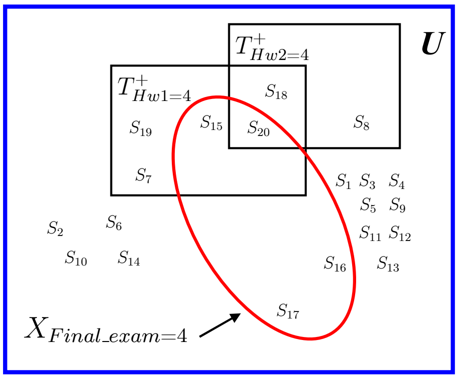


Fig. 6. Evaluation of successful final exams with positive tools

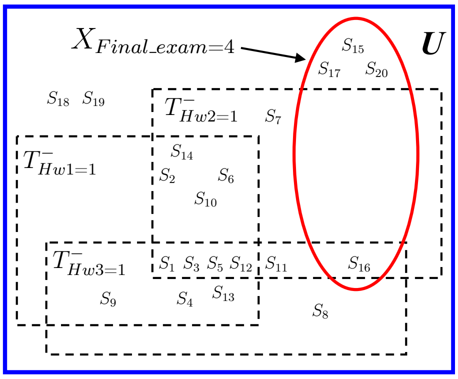


Fig. 7. Evaluation of successful final exams with negative tools

Students who have failed their final exams can also be evaluated with both positive and negative tools (Fig. 8, Fig. 9):

- $\mathfrak{C}^{\flat}_{\mathfrak{T}^{+}}(X_{Final\_exam=1}) = \emptyset$
  Informally: there is no combination of successful homework in which case the final exam *surely* fails.
- $\mathfrak{C}^{\sharp}_{\mathfrak{T}^{+}}(X_{Final\_exam=1}) = T^{+}_{Hw1=4}$
  Informally: if the only Homework 1 succeeds, the final exam *possibly* fails (because, e.g., Homework 1 is the simplest part of the course).
- $\mathfrak{C}^{\flat}_{\mathfrak{T}^{-}}(X_{Final\_exam=1}) = T^{-}_{Hw1=1 \wedge Hw2=1 \wedge Hw3=1}$
  Informally: If all homework fail, the final exam *surely* fails.
- $\mathfrak{C}^{\sharp}_{\mathfrak{T}^{-}}(X_{Final\_exam=1}) = \bigcup \mathfrak{T}^{-}$
  Informally: if at least one homework fails, the final exam *possibly* fails.

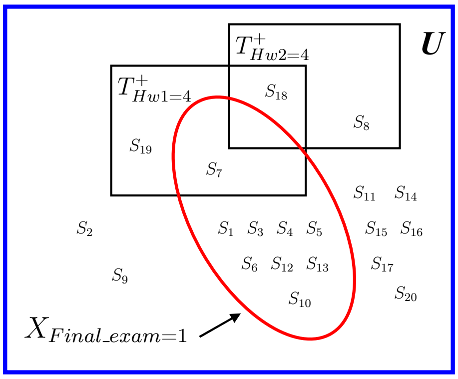


Fig. 8. Evaluation of failed final exams with positive tools

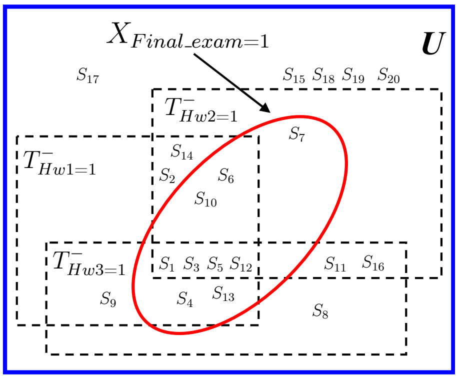
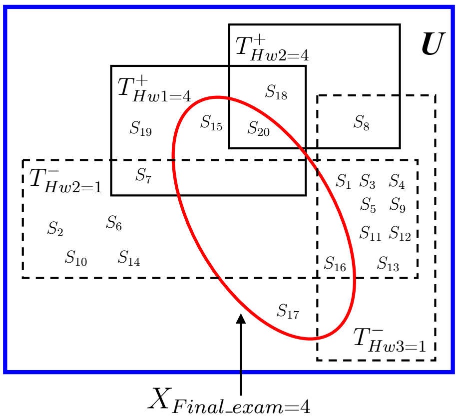


Fig. 9. Evaluation of failed final exams with negative tools

Evaluations can also be carried out over positive and negative tools together:

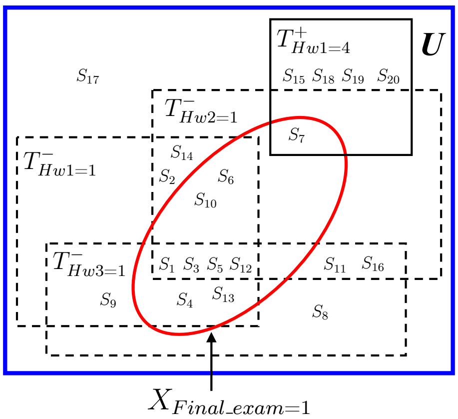
– $\mathfrak{C}^{\flat}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=4}) = \emptyset$ (see Fig. 10) informally means that there is no combination of successful or failed homework in which case the final exam *surely* succeeds.
– $\mathfrak{C}^{\sharp}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=4})$ (see Fig. 10) informally means that if one of the Homework 1, 2 or both of the two succeed, in addition, even if one of the Homework 1, 3 or both of the two fail, then the final exam *possibly* succeed.
– $\mathfrak{C}^{\flat}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=1})$ (see Fig. 11) informally means that if at least one homework fails, the final exam *surely* fails.
– $\mathfrak{C}^{\sharp}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=1})$ (see Fig. 11) informally means that if at least one homework fails, the final exam *possibly* fails even if the Homework 1 succeeds.



$$\mathfrak{C}^{\flat}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=4}) = \emptyset$$

$$\begin{aligned} &\mathfrak{C}^{\sharp}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=4}) \\ &= T^+_{Hw1=4} \cup T^+_{Hw2=4} \\ &\quad \cup T^+_{Hw1=4 \wedge Hw2=4} \\ &\quad \cup T^-_{Hw2=1} \cup T^-_{Hw3=1} \\ &\quad \cup T^-_{Hw2=1 \wedge Hw3=1} \end{aligned}$$

Fig. 10. Evaluation of successful final exams with positive and negative tools



$$\begin{aligned} &\mathfrak{C}^{\flat}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=1}) \\ &= T^-_{Hw1=1 \wedge Hw2=1 \wedge Hw3=1} \end{aligned}$$

$$\begin{aligned} &\mathfrak{C}^{\sharp}_{\mathfrak{T}^+ \cup \mathfrak{T}^-}(X_{Final\_exam=1}) \\ &= T^-_{Hw1=1} \cup T^-_{Hw2=1} \cup T^-_{Hw3=1} \\ &\quad \cup T^-_{Hw1=1 \wedge Hw2=1} \\ &\quad \cup T^-_{Hw1=1 \wedge Hw3=1} \\ &\quad \cup T^-_{Hw2=1 \wedge Hw3=1} \\ &\quad \cup T^-_{Hw1=1 \wedge Hw2=1 \wedge Hw3=1} \\ &\quad \cup T^+_{Hw1=4} \end{aligned}$$

Fig. 11. Evaluation of failed final exams with positive and negative tools

6 Conclusion

We have presented in this paper a tool-based set theoretic framework for concept approximation relying on partial approximation spaces. Positive features and their substantially negative features of observed objects can *simultaneously* be approximated with the help of this framework.

We have drawn up a simplified example to demonstrate our approach. We have analyzed a student grade history and we have been able to evaluate the students' achievement, exploring 'what they know' and/or 'what they do not know', and understand how the results in homework assignments approximately relate to success or failure on the final exam. Of course, a more subtle definition of the notions of 'success' and 'failure' could result in a more subtle evaluation. A refined evaluation process can form a basis for *quality insurance in higher education* properly building in the hierarchy of quality management.

Acknowledgement

The author would like to express his gratitude to the anonymous referees for reading the paper carefully and their insightful comments and suggestions.

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