

Mathematical Modeling of Formation Processes of Sequences with Fractal Elements of Periodical Chaotic Dynamical System Trajectories

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Abstract. This paper considers problems that arise during number sequence generation based on nonlinear dynamical systems. Complex systems can depend on many parameters analysis and examination of one-dimensional maps was performed since these maps are dynamical systems. Dependence of iterative fixed points for nonlinear maps on the properties of functions and function domain numbers was investigated. Several approaches to randomness evaluation and, accordingly, methods for estimating the degree of randomness of a particular sequence were considered. The properties and internal structure of sequences obtained on the basis of nonlinear maps were also examined in accordance to their influence on the degree of randomness.

Keywords. Chaos, pseudorandom sequences, nonlinear maps, prime numbers.

1 Introduction

The current state of development of information technology creates the illusion that existing methods of its application give opportunity to humanity to solve any problem of complex dynamical system management for any level of complexity. This is facilitated by the well-developed idea that problems and solving technologies for them do not largely depend on the level of development of the mathematical foundations of information theory, however in reality the number of blind spots in the theory of application of novel methods only increases, since current discoveries raise more and more new questions. In this regard, the development of the theory of dynamical systems is more relevant than ever [1].

The task of studying the structure of trajectories of cyclic fixed points of nonlinear dynamical systems is assigned to the so-called blind spots. According to Sharkovskii's theorem [2] if a discrete dynamical system on the real line has a periodic point of period 3, then it must have periodic points of every other period. In general, this statement determines that it leads to the creation of so-called chaos.

In the chaotic behavior of dynamical systems, the slightest inaccuracy in determining the initial state of the system increases rapidly over time and, therefore, forecasting becomes ineffective or even impossible. Understanding of chaos becomes one of

the most important problems of our time. Solution of this problem leads us to attempts to understand all the variety of nonlinear phenomena and processes in real-world aspects [3, 15].

Due to the dynamical nature of chaotic modes and their sensitivity to the slightest oscillations, they provide effective control by external controlled influence. The purpose of such influence can be realization of periodic mode in system instead of chaos or getting to the given phase space [6,7].

The purpose of this work is to create a mathematical basis for information technology analysis of structures of cyclic trajectories of complex dynamical systems, as a way to determine chaos in nonlinear stochastic dynamical systems of varying complexity.

It is especially important to identify the structure of chaotic processes of trajectories of cyclic fixed points over a large length, which are determined by numbers in form $1/p$, where p is a large prime number.

The analysis of such trajectories is important for the reason that the structures of trajectories for n/m values for fixed points can be analytically represented through the properties of trajectories determined by prime numbers using functional maps formed on the basis of group theory [8, 9].

In this way, it will be determined whether the obtained trajectories have a deterministic structure and how it affects the structure on the set of integers, rational or real numbers. The dynamical properties of cyclic trajectories can be thoroughly investigated only by transformation of mathematical models into a recursive form. The theory of recursive functions allows to make the transition to representation of given models in recursive form [10].

Mathematical theory of iterative maps in the general case refers to the theory of recursive functions, the theory of dynamical systems, the theory of differential equations and is deeply related to modern chaos theory [11]. Dynamical systems use iterations of the form $x_n \rightarrow x_{n+1} = f(x_n)$, where the function $f(x)$ is defined on the interval $[0; 1]$ and translates into itself and belongs to the class of primitive recursive functions [10,12]

An important example of such systems is the process of population development in an environment with certain properties [16]. The mathematical model of the stochastic dynamical system of population development can be presented in Wright-Fisher form:

$$dx_{\xi}(t) = c \cdot \sum_{\xi' \in S} q_{\xi' \xi} (x_{\xi'}(t) - x_{\xi}(t)) dt + s \cdot x_{\xi}(t) (1 - x_{\xi}(t)) dt + \sqrt{\gamma x_{\xi}(t) (1 - x_{\xi}(t))} d\omega_{\xi}(t), \quad x_{\xi}(0) \in [0,1] \quad \forall \xi \in S \quad (1)$$

This equation is reduced to an iterative form which has the following form.

$$x_{\xi}(n+1) = x_{\xi}(n) + c \cdot \sum_{\xi' \in S} q(i, j) (x_{\xi'}(n) - x_{\xi}(n)) + s \cdot x_{\xi}(n) (1 - x_{\xi}(n)) + \gamma \sqrt{x_{\xi}(n) (1 - x_{\xi}(n))} \omega(n) \quad (2)$$

The obtained iterative equation has a complex form. The behavior of such a dynamical system largely depends on the values of the constants $c, s, \gamma, S, q(i, j)$, of the environment properties $e_{x_i}(t)$, characteristics of the change in the genetic code of the population $\omega_i(t)$ and the magnitude of its drift.

It should be noted that dynamical systems given by different mathematical maps sometimes allow to obtain the same information. Another class of problems that is related to the development of estimation methods of fractal measure for sets, functions, random processes and the formation of information in systems and control methods for dynamical systems based on information on the structure of their cyclic trajectories.

2 Structural analysis methods for cyclical trajectories of nonlinear dynamical systems

The results of the study of dynamical systems depend on the properties of parameters that characterize these systems. Systems can depend on many parameters, but the results of analysis and research of dynamical systems based on one-dimensional maps can be applied to a wide range of complex dynamical systems.

Consider the maps of the following classes: «Tent», «Asymmetric tent», «Discontinuous tent», «Sawtooth», "Logistic map", «Square root logistic map», «Sine map», "Algebraic map". This choice of maps is subjected to the fact that some classes of complex maps can be considered as functions of certain combinations of these maps. In addition, there are relationships between these maps that are a source of important information about the relationships between classes of primes used in discrete logarithm theory, in methods for constructing efficient pseudorandom number generators and in number theory [13, 14]. The abovementioned maps are considered as a function $f(x): \mathbb{R} \rightarrow \mathbb{R}$.

Then $f^n(x)$ denotes the n th iteration of the function $f(x)$. Then $f^n(x)$ is n -fold composition of the function $f(x)$ with itself. If $x_0 \in \mathbb{R}$, then the orbit or trajectory for x_0 is a sequence $x_0, x_1 = f(x_0), \dots, x_n = f^n(x_0), \dots$

When dynamical systems are considered, fixed points, cyclic fixed points and their trajectories play an important role. Initial point x_0 is fixed by condition $f(x_0) = x_0$. Based on this, we can conclude that the orbit of a fixed point is a constant sequence x_0, x_0, x_0, \dots . These are such values x_0 that $f^n(x_0) = x_0, n > 0$, accordingly, like closed orbits, periodic orbits are repeated: $x_0, \dots, x_{n-1}, x_0, \dots$. Hereinafter, the periodic orbits will be called trajectories. For discrete dynamical systems, several types of fixed points can be distinguished. Suppose that x_0 is a fixed point for f , then x_0 called an attractor or an attractive fixed point for f . If there is a neighborhood U of a point x_0 on the set \mathbb{R} with such a property that, if the following conditions are met: $y_0 \in U$, then $f^n(y_0) \in U$ for all n and, moreover, $f^n(y_0) \rightarrow x_0, n \rightarrow \infty$. Similarly, x_0 is defined as a repellent or repulsive fixed point if all orbits (except x_0) leave U during iterations of function $f(x)$.

Consider the symmetric «Tent» map $g_1(x)$ and the algebraic map $alg(x)$ that calculates residue modulo prime on the basis of a that is greater than 1 and initial value $x_0 = 1/p$ that defines behavior of given maps.

$$g_1(x_n) = \begin{cases} 2x_n, & x_n < 1/4 \\ 1 - 2x_n, & 1/4 \leq x_n < 1/2 \end{cases} \quad (3)$$

$$alg(x_n) \equiv a \cdot x_n \pmod{p} \quad (4)$$

In [8] it was proved that a number a under this condition is a classifier of the set of all prime numbers. Assuming that $card_{g_1}(1/p)$ represents the cyclic trajectory length for a prime number p for map $g_1(x)$ the following statement holds.

Statement 1. For any prime number p value of $p - 1$ is always divisible by $card_{g_1}(p)$ and its value $(p - 1) / card_{g_1}(p)$ always coincides with $ind_a(p)$ for an algebraic map under the condition that $a = 4$.

The validity of this statement is proved by the method of computer modeling, the general principle of its construction is given in [10,11]. It follows that symmetric tent map can be used to analyze the properties of prime numbers. At the same time, the results of the same computer modeling show that the trajectories of algebraic map and symmetric tent map do not coincide in their structure in case if they are reduced to the same scale of values along the ordinate axis.

Although the map «Sawtooth» $g_2(x)$ is considered one of the simplest examples of a nonlinear dynamical system, it demonstrates an important property that is inherent in more complex algebraic maps with $a = 2$ on prime numbers set and initial value $x_0 = 1/p$.

$$g_2(x_n) = \begin{cases} 2x_n, & x_n < 1/2 \\ 2x_n - 1, & 1/2 \leq x_n < 1 \end{cases} \quad (5)$$

Similar to statement 1, if $card_{g_2}(p)$ is a trajectory length for map $g_2(x)$ and $x_0 = 1/p$, then the following statement is valid.

Statement 2. For any prime number $p > 3$ value of $p - 1$ is always divisible by $card_{g_2}(p)$ and its value $(p - 1) / card_{g_2}(p)$ coincides with $ind_a(p)$ for algebraic map with condition $a = 2$.

The validity of this statement immediately follows from the fact that the value of the numerator of the fraction reflected in each iteration coincides with the value of the corresponding iteration of the algebraic map $g_2(x)$. Thus, the map $g_2(x)$ can be used to analyze the classification of prime numbers, the construction of generators and complex dynamical system modeling [9].

Consider a map $g_3(x)$ that has a point of discontinuity, i.e. it can be considered as discontinuous tent at $x = 1/2$. At the point of discontinuity $g_3(1/2) = 1/2$ in this model, but the discontinuity can take place at another x values. Initial value for this map is also $x_0 = 1/p$.

$$g_3(x_n) = \begin{cases} 2x_n, & x_n < 1/2 \\ 1 - x_n, & 1 > x_n \geq 1/2 \end{cases} \quad (6)$$

It was proved that on a set of numbers $x_0 = 1/p$ ginen map has cyclic trajectories with length $\text{card}_{g_3}(1/p)$ that in general case does not divide $(p - 1)$ despite the fact that the map can be classified as tent map modification (discontinuous tent). The dynamical properties of this map indicate that the lack of information about the properties of cyclic trajectories can lead to its incorrect use in the approximation of complex dynamical systems. This fact suggests that in dynamical system modeling it is necessary to have accurate information about the properties of the components of mathematical models.

Behavioral study of cyclic trajectories of tent map class fixed points is an equally important task in process of fundamental theory of complex dynamical systems creation. A complete study of the attractors of this class of maps was performed in various researches. However, there is no information about the structure of their trajectories. It is necessary to create methods that would allow under any circumstances to have detailed information about trajectory properties.

Asymmetric tent.

An example of a map that belongs to the tent class is an asymmetric map. It can be considered as an extension of the standard «Tent»:

$$f(x_n) = \begin{cases} \frac{x_n}{a}, & 0 \leq x_n \leq a \\ \frac{1 - x_n}{1 - a}, & a \leq x_n \leq 1 \end{cases} \quad (7)$$

The chaotic behavior for this map is shown in Fig. 1 that represents partial trajectory of the map with arbitrarily selected prime number. The essence of chaos in this case is that the values periodically approach each other, but never repeat themselves.

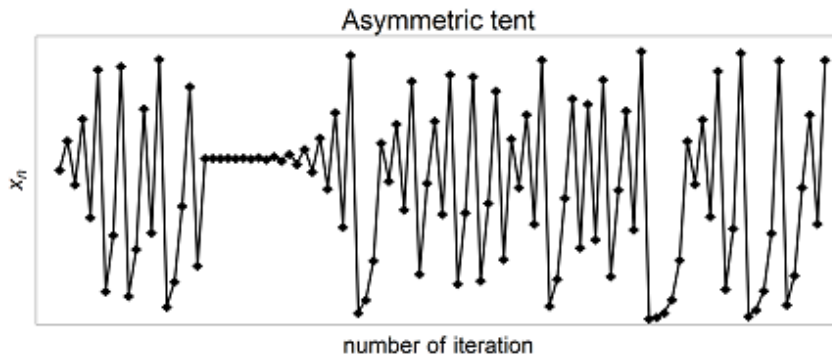


Fig. 1. Chaotic structure of asymmetric tent

Logistic map.

The logistic map is defined as follows:

$$f(x_n) = rx_n(1 - x_n), \quad (8)$$

where $r \geq 0$ is the parameter that fixes population growth. Logistic map demonstrates the properties of nonlinear dynamical systems that simulate a wide class of dynamical processes that belong to the class of population development. It should be noted that processes occurring at currency and stock markets, actuarial mathematics systems and others can be effectively modeled using logistic map. The choice of this map as a component of the mathematical model is associated with the problem of choosing the value of the parameter r . It is proved that if the value of the parameter is less than 3.6, then the logistic map simulates the processes of sustainable development of dynamics and for larger values the behavior of trajectories acquires more chaotic nature. Especially chaotic character of cyclic trajectories is observed when approaching $r = 4$. Under such conditions, it becomes extremely important to have information about the behavior of a dynamical system when x variable takes the value of the form $1/p$ where p is a prime number of significant values. The properties of trajectories in such cases have not been studied. At least in the literature, such information is absent, although logistic map is the subject to a wide range of studies.

Sine map.

The map $f(x) = \sin(\pi x)$ is very similar in structure to the logistic map, it should be noted that this function usually takes values from the interval $[0,1]$ and this function is associated with a rather complex group that is not fully analyzed. In addition, this function usually takes irrational values, thus in modeling of cyclic trajectories intermediate values of the trajectories will be rounded systematically, which must be taken into account due to the fact that it will lead to the transition to another state of the dynamical system.

Square root logistic map.

This map is a modification of logistic map and can be represented in next form:

$$f(x) = \sqrt{rx_n(1 - x_n)} \quad (9)$$

As a rule, it is used in modeling of slow processes of genetic changes in dynamical systems. This is especially true for stock markets and the processes of genetic code change for developing populations, such as coronaviruses. It should be noted that in such models the constant a takes values much smaller than 3.6 because the processes of genetics slow change of such dynamical systems are modeled. In this case, the strategy of resource use of the environment is systematically adapted to changes in its resources, and therefore the study of cyclic trajectories of such maps is an important task.

2.1 Congruence measure for given maps

Provided one-dimensional maps demonstrate similar behavior of the corresponding nonlinear dynamical systems. All these maps show so-called chaotic behavior since they are sensitive to the initial conditions. As a rule, the given maps are considered on the interval $[0,1]$, taking into account that the obtained results are easily transferred to the intervals of any length $[1, n]$, taking into account the significant similarity of their structures except for algebraic map. Algebraic map is included in the work due to the fact that it is fundamental in modern number theory and pseudorandom number generation methods and modern cryptography. Framework of analysis of the congruence of different maps is the proof that between the cyclic fixed points there is an unambiguous correspondence at which the congruent fixed points have the same length. Algebraic map can be seen as a connecting link between function theory and number theory.

Consider “Tent” map:

$$f_1(x_n) = \begin{cases} 2x_n, & x_n < \frac{1}{4} \\ 1 - 2x_n, & 1 > x_n \geq \frac{1}{4} \end{cases} \quad (10)$$

Particular attention to map (10) draws the fact that it shares many properties with the logistic map (8) when $r = 4$ in the iteration process. This feature indicates their conjugation. Assuming that I and J represent some intervals for maps $f: I \rightarrow I, g: J \rightarrow J$, then it could be assumed that the maps f and g conjugates, if there is a homeomorphism $h: I \rightarrow J$, still h satisfies the equality of conjugation $h \circ f = g \circ h$. Conjugacy compares orbits f to orbits g . This follows from the fact that $h(f^n(x)) = g^n(h(x))$ for all $x \in I$ such that h compares the n -th point of the orbit for f from x to the n -th point of the orbit g from $h(x)$.

Statement 3. Tent map is congruent with logistic map, square root logistic map, and sine map.

The proof is based on the following sequence of steps that follow from the work [4]. The validity of the fact that sine map is congruent to the logistic map follows from the following simple transformations. Let's define the equation of conjugation by analogy of definition of logistic map at $a = 4$:

$$h(2x) = 4h(x)(1 - h(x)) \quad (11)$$

for $x \in [0, \frac{1}{2}]$. Note that the above satisfies the function of the form

$$h(x) = \sin^2(cx) \quad (12)$$

for any $c > 0$ by the formula: $\sin(2\theta) = 2 \sin \theta \cos \theta$.

In order h to be a homeomorphism of the interval itself, it must be monotonic on any interval. Closer examination of homeomorphism defined by formula (12) allows

to consider $c = \frac{\pi}{2}$. It follows that $h(0) = 0$, $h(1) = 1$ and h is increasing throughout the interval $[0, 1]$.

To make sure that (12) is really a topological conjugation, it is necessary to examine it on the interval $x \in [\frac{1}{2}, 1]$:

$$h(2(1-x)) = 4h(x)(1-h(x)) \quad (13)$$

Let's perform substitution and simplification:

$$\begin{aligned} h(2(1-x)) &= \sin^2(\pi(1-x)) = \sin^2(\pi x) = \\ &= 4 \sin^2\left(\frac{\pi x}{2}\right) \cos^2\left(\frac{\pi x}{2}\right) = 4h(x)(1-h(x)) \end{aligned} \quad (14)$$

That was necessary to prove.

Proved statement shows that sine map can be viewed as an example of a topologically conjugate system with logistic map and tent map. From this theorem follows only the fact that in congruent cyclic fixed points the maps have the same lengths of cyclic trajectories, but their structures can be different to a large extent. The values of these maps belong to different classes of numbers. The values of the cyclic trajectories of tent and logistic maps take rational values; square root logistic map belongs to the class of algebraic numbers and sine function is defined on the set of real numbers which in the general case are transcendental. These facts must be taken into account during modeling of the behavior of such dynamical systems due to the fact that automatic rounding of values can lead to an automatic transition to completely different cyclic trajectories [5].

2.2 Analysis of cyclic trajectories structure of map fixed points

To solve many applied problems, for example, design of pseudorandom sequence generators the theory of residues modulo large prime with basis a which is primitive root of the selected p is used. In this case, based on Fermat's theory, the equation $a^{p-1} \equiv 1 \pmod{p}$ is the basis of the iterative procedure, which has already been defined in this paper as $alg(x_n)$. Using a defined iterative process, a sequence of values is obtained in the form of a set $\{x_0, x_1, \dots, x_n, \dots, x_{p-1} = 1\}$, which is interpreted as a pseudorandom sequence. Based on statistical analysis, it is proved that for large values of p obtained sequences for all primitive roots $\{a, \dots, a_{\varphi(p-1)}\}$, where $\varphi(p-1)$ is Euler function, provide wide range of variants with similar property that can be interpreted as chaos.

The map $f(x) \equiv a \cdot x_n \pmod{p}$ is obviously related to the group of residues $(\mathbb{Z}/p\mathbb{Z})^*$ by modulo p . If a is primitive root of a prime number p , it is easy to prove that a is the generating element of this group and the iterative process determines the permutation on the set $\{1, 2, 3, 4, \dots, p-1\}$ which forms a cyclic subgroup of permutations of the order p of the complete permutation group on this set. This simple fact is an important basis for arguing that such a group cannot have a chaotic structure. This

statement can be demonstrated by the example when $p = 19$, and all six primitive roots $\{2, 3, 10, 13, 14, 15\}$ are considered and $\varphi(19 - 1) = \varphi(2 \cdot 3^2)$. For this set of primitive roots based on the iterative procedure, the following permutations are obtained and shown in Table 1:

Table 1. Permutations for the set of primitive roots

α iteration	2	3	10	13	14	15
1	2	3	10	13	14	15
2	4	9	5	17	6	16
3	8	8	12	12	8	12
4	16	5	6	4	11	9
5	13	15	3	14	10	2
6	7	7	11	11	7	11
7	14	2	15	10	3	13
8	9	6	17	16	4	5
9	18	18	18	18	18	18
10	17	16	9	6	5	4
11	15	10	14	2	13	3
12	11	11	7	7	11	7
13	3	14	13	15	2	10
14	6	4	16	5	9	17
15	12	12	8	8	12	8
16	5	17	4	9	16	6
17	10	13	2	3	15	14
18	1	1	1	1	1	1

Thus, if recursive sequence for a primitive root is given then all other primitive roots can be computed. Above stated findings follow from cyclic groups theory. From the given example it follows that for $p = 19$ clear structure of permutations emerges. It is proved that for any prime number p of arbitrarily larger value, the structure determined by the expansion into prime factors of the number $p - 1 = \prod_{i=1}^k p_i^{\alpha_i}$.

Thus, the above sequence has a clear structure, which does not allow us to consider it as a chaotic system.

2.3 1-D map structure analysis

Despite the simplicity of these maps their structure largely depends not only on the properties of the functions used for their construction, but also on the properties of the numbers used as initial conditions and parameters. These maps allow you to divide the set of primes into a system of classes based on the length of the iterative process for given primes. Note that there are many prime numbers for which the length of the period is significantly less than the dimension of the number. The sequence obtained for a given number forms a simple structure. Simple structures are characteristic of Fermat, Mersenne numbers and their various generalizations. At the same time, other

prime numbers generate sequences for which the length of the period is proportional to its dimension and, accordingly, can show a greater degree of approximation to randomness, but also have periodic elements as shown in Figures 2-4.

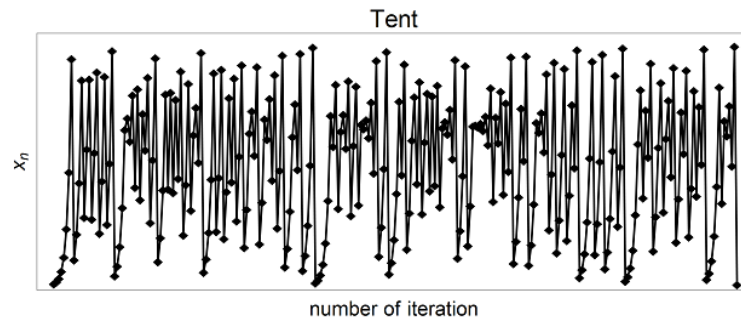


Fig. 2. "Tent" map iteration sequence

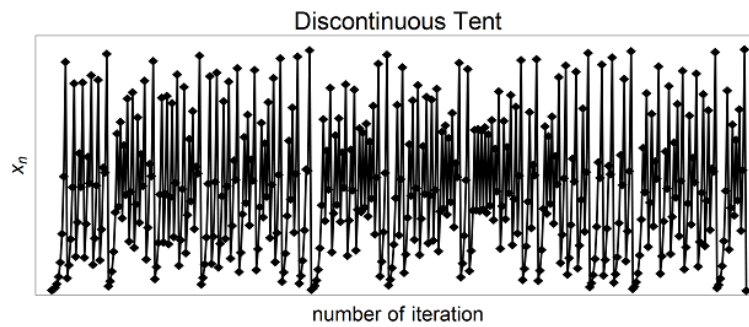


Fig. 3. "Discontinuous Tent" map iteration sequence

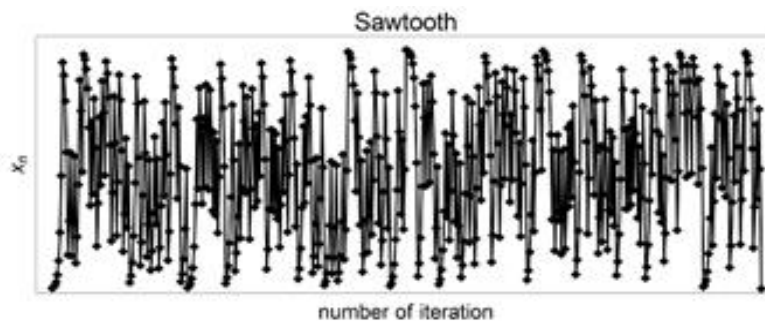


Fig. 4. "Sawtooth" map iteration sequence

The question of a certain degree of similarity of internal structures arises. For example, for the prime number 649657, Figure 5 shows the internal structure of iterative processes for maps, where the dotted line shows the resulting sequences and

the solid line shows the internal parts within the sequences that give the maximum value of the correlation coefficient.

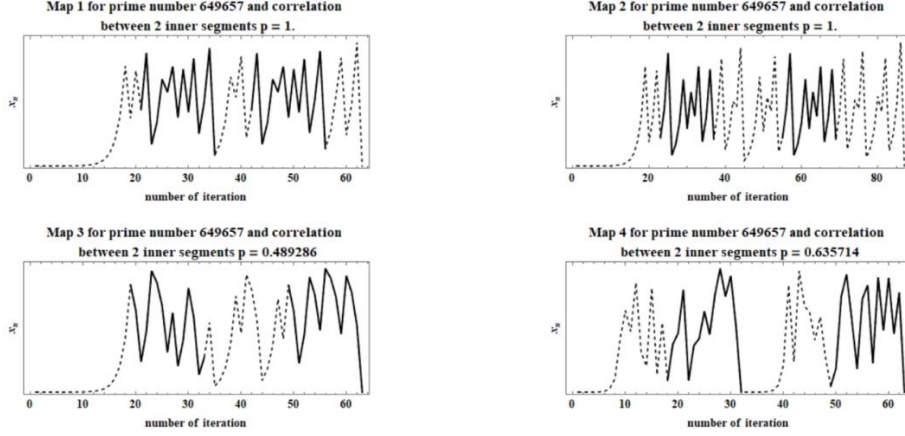


Fig. 5. Structure of sequence on the basis of prime number 649657

As can be seen in these sequences, which were obtained using the above maps for some subsequences, give similarity values closer to 0, indicating the influence of fixed points on the internal structure of the sequence

2.4 Estimation methods for measure of randomness

There are several approaches to determining randomness and, accordingly, methods for estimating the degree of randomness of a particular sequence. Four algorithmic properties differ for the description of randomness: frequency stability, chaos, typicality, unpredictability. Each of them represents its own algorithmic aspect of randomness and each of them, with more or less feature, can claim a mathematical definition of the concept of randomness.

When considering the internal structure of numerical sequences, the presence of internal similar subsequences means that these internal structures can be grouped into separate classes and for each class you can assign a description that will reduce the size of the description of the whole sequence. Thus, when considering the internal structure of the formed sequences, the closest to the chaotic will be those that will have the least degree of similarity.

The simplest measure of similarity for comparing subsequences is any norm L_n of the form:

$$d_{L_n}(x, y) = \left(\sum_{i=1}^M (x_i - y_i)^n \right)^{1/n}, \quad (15)$$

where n is an integer, M is the length of the subsequence. Measures based on norms L_n belong to the category of rigid step-by-step measures and compare structures of the same length. In the case when $n = 2$ we obtain the Euclidean norm, the evaluation process of which is shown in Figure 6.

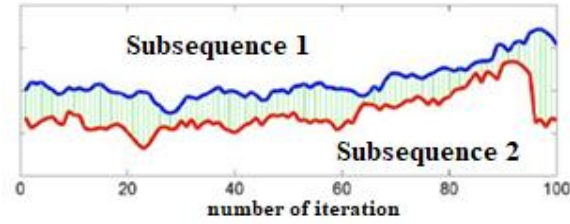


Fig. 6. Process of sequence comparison

However, such measures do not identify the similarity of subsequences if they are not aligned according to all X . Accordingly, there is a problem of "deformation" of the values of all X for one of the subsequences. This problem allows to solve elastic measures, such as the method of dynamical scale transformation (DST) [12], but such measures increase the complexity of calculations and the time required to obtain the result. When determining the DST measure, the local cost matrix (LC) ($n \times m$) is first calculated where each element of the matrix determines the distance between the corresponding elements of the subsequences. The next step is to determine the path of transformation:

$$W = w_1, w_2, \dots, w_k, \quad \min(n, m) \leq k \leq m + n + 1 \quad (16)$$

This path bypasses the LC matrix with such conditions as: boundary condition, discontinuity, monotonicity. The total distance for the path W is determined by summing the individual elements of the LC matrix that cover the path. To obtain the DST measure, it is necessary to choose a path with a minimum total distance. The complexity of the calculation in this case is $O(nm)$ when using the methods of dynamical programming (DP). The following recurrent DP ratio can be used to calculate the path with the minimum length:

$$d_{cum}(i, j) = d(x_i, y_j) + \{d_{cum}(i-1, j-1), d_{cum}(i-1, j), d_{cum}(i, j-1)\} \quad (17)$$

Several scales have been developed to obtain the DST measure directly, one of which is the root of the sum of the path elements with the minimum length:

$$d_{DTW}(x, y) = \sqrt{\sum_{k=1}^K w_k} \quad (18)$$

It should be noted that the DST measure is equal to the Euclidean norm, if $n = m$. Due to the need to calculate matrices, this method is one of the most time-consuming, even under optimization, so it is not considered in this paper.

The next group of measures of similarity are characteristic measures, the main of which is the discrete Fourier transform (DFT). As mentioned above, this measure evaluates the characteristics of the compared structures and, since it is calculated for only half of the elements of the subsequence $q = \frac{n}{2}$ according to the Nyquist-Shannon calculus, allows to obtain a gain in the total computation time. DFT is obtained by calculating the product between a subsequence and a sine wave and is defined as:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \quad (19)$$

This paper considers the application of the form estimation measure and the DFT measure as an example of a characteristic measure, as they provide a simple process for computing and allow conclusions to be drawn about the internal structure of the sequences considered in this paper. The measure based on the correlation coefficient was chosen as a rigid stepwise measure. To assess the internal structure of the sequences obtained on the basis of maps, a method is used that includes the following steps:

1. 1. The position of the first peak is calculated in order to remove the initial exponential component from consideration;
2. 2. Determine the size of the initial subsequence for evaluation with the following elements of the sequence;
3. 3. Using a single step, calculate the value of the Spear-correlation of the reference subsequence with the corresponding size subsequences;
4. 4. The obtained correlation values are filtered according to the specified level of similarity;
5. 5. The size of the initial subsequence is reduced by 1 if it exceeds 10 elements, and steps 1-4 are repeated.

Comparing the selected similarity measures, the best results from the search for similar subsequences are demonstrated by the DFT measure, as it represents the estimation of the subsequence in the form of the sum of harmonic oscillations, respectively, allows a more accurate estimation. The time complexity for DFT when using the fast Fourier transform algorithm is $O(n \log n)$.

3 Conclusion

Analysis of various approaches to the construction of pseudorandom sequences revealed that choice of initial conditions affects the structure of obtained sequence: if the length of the sequence length corresponds to the dimension of the selected prime, then such a sequence is more similar to chaotic. It was found that under any

circumstances, the existence of fixed points in the selected nonlinear dynamical system does not lead to chaos and on the contrary unambiguously determines the existence of a stable structure with a fixed cycle length. The largest number of internal cycles is demonstrated by sequences based on «Discontinuous tent» map and it follows that this map generates more deterministic sequences than the other considered maps.

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