Triadic Factor Analysis

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Abstract. This article is an extension of work which suggests using formal concepts as optimal factors of Factor Analysis. They discussed a method for decomposing a $p \times q$ binary matrix W into the Boolean matrix product $P \circ Q$ of a $p \times n$ binary matrix P and a $n \times q$ binary matrix Q with n as small as possible.
We have generalised this factorization problem to the triadic case, looking for a decomposition of a $p \times q \times r$ Boolean $3d$-matrix $\mathbb{B}$ into the Boolean $3d$-matrix product $P \circ Q \circ R$ for $p \times n, q \times n$ and $r \times n$ binary matrices P, Q and R with n as small as possible.
The motivation is given by the increasing interest in Triadic Concept Analysis due to Web 2.0 applications.

Keywords: Triadic Concept Analysis, Factor Analysis, Web 2.0

1 Introduction and Previous works

The interest in decomposition of triadic data in psychology goes back at least to the beginning of the 90's [2]. However at that time the methodologies were not as developed as nowadays. The interest in triadic data is also increasing due to the structure of Web 2.0 data [4].

This paper contains a new method for factorization of triadic data which could be developed as a generalisation of the dyadic case presented in [1].[1]

2 Triadic Concept Analysis

For an introduction to *Formal Concept Analysis* we refer the reader to [5]. The triadic approach to Formal Concept Analysis was introduced by Rudolf Wille and Fritz Lehmann in [6]. It is based upon C.S. Pierce's pragmatic philosophy with his three universal categories. We just give a compressed introduction to this field and refer the interested reader to [6, 7].

Definition 1. *A* ***triadic context*** *is defined as a quadruple* (K_1, K_2, K_3, Y) *where* K_1, K_2 *and* K_3 *are sets and* Y *is a ternary relation between* K_1, K_2 *and*

[1] During the revision period it turned out there is yet unpublished work of R. Belohlavek and V. Vychodil dealing with the same subject [3]

K_3, i.e. $Y \subseteq K_1 \times K_2 \times K_3$. The elements of K_1, K_2 and K_3 are called ***(formal) objects, attributes*** *and* ***conditions****, respectively, and $(g, m, b) \in Y$ is read: the object g has the attribute m under the condition b.*

Small triadic contexts can be represented through three-dimensional cross tables. Examples are given in Table 1 and 2 on page 6 and 11.

Definition 2. *A* ***triadic concept*** *(shortly* ***triconcept****) of a triadic context (K_1, K_2, K_3, Y) is defined as a triple (A_1, A_2, A_3) with $A_1 \subseteq K_1$, $A_2 \subseteq K_2$ and $A_3 \subseteq K_3$ that is maximal with respect to component-wise set inclusion in satisfying $A_1 \times A_2 \times A_3 \subseteq Y$, i.e. for $X_1 \subseteq K_1$, $X_2 \subseteq K_2$ and $X_3 \subseteq K_3$ with $X_1 \times X_2 \times X_3 \subseteq Y$, the containments $A_1 \subseteq X_1, A_2 \subseteq X_2$, and $A_3 \subseteq X_3$ always imply $(A_1, A_2, A_3) = (X_1, X_2, X_3)$.*

For a triconcept (A_1, A_2, A_3), the components A_1, A_2 and A_3 are called the ***extent****, the* ***intent****, and the* ***modus*** *of (A_1, A_2, A_3) respectively.*

Pictorially, a triconcept (A_1, A_2, A_3) is a rectangular box full of crosses in the three-dimensional cross table representation of (K_1, K_2, K_3, Y), where 'box' is maximal under proper permutation of rows, columns and layers of the cross table.

Definition 3. *For $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$ and for $X \subseteq K_i$ and $Z \subseteq K_j \times K_k$, the (i)-****derivation operators*** *are defined by:*

$$X \mapsto X^{(i)} := \{(a_j, a_k) \in K_j \times K_k \mid (a_i, a_j, a_k) \in Y \text{ for all } a_i \in X\},$$
$$Z \mapsto Z^{(i)} := \{a_i \in K_i \mid (a_i, a_j, a_k) \in Y \text{ for all } (a_j, a_k) \in Z\}.$$

These derivation operators correspond to the derivation operators of the dyadic contexts defined by $\mathbb{K}^{(1)} := (K_1, K_2 \times K_3, Y^{(1)}), \mathbb{K}^{(2)} := (K_2, K_1 \times K_3, Y^{(2)})$ and $\mathbb{K}^{(3)} := (K_3, K_1 \times K_2, Y^{(3)})$ where $gY^{(1)}(m, b) :\Leftrightarrow mY^{(2)}(g, b) :\Leftrightarrow bY^{(3)}(g, m) :\Leftrightarrow (g, m, b) \in Y$. Due to the structures of triadic contexts further derivation operators are necessary for the computation of triconcepts. For $\{i, j, k\} = \{1, 2, 3\}$ and for $X_i \subseteq K_i$, $X_j \subseteq K_j$ and $X_k \subseteq K_k$ the (i, j, X_k)-**derivation operators** are defined by

$$X_i \mapsto X_i^{(i,j,X_k)} := \{a_j \in K_j \mid (a_i, a_j, a_k) \in Y \text{ for all } (a_i, a_k) \in X_i \times X_k\},$$
$$X_j \mapsto X_j^{(i,j,X_k)} := \{a_i \in K_i \mid (a_i, a_j, a_k) \in Y \text{ for all } (a_j, a_k) \in X_j \times X_k\}.$$

These derivation operators correspond to the derivation operators of the dyadic contexts defined by $\mathbb{K}_{X_k}^{ij} := (K_i, K_j, Y_{X_k}^{ij})$ where $(a_i, a_j) \in Y_{X_k}^{ij}$ if and only if $(a_i, a_j, a_k) \in Y$ for all $a_k \in X_k$.

The structure on the set of all triconcepts $\mathfrak{T}(\mathbb{K})$ is the set inclusion in each component of the triconcept. There is for each $i \in \{1, 2, 3\}$ a quasiorder $\lesssim_i$ and its corresponding equivalence relation $\sim_i$ defined by

$$(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3) :\Longleftrightarrow A_i \subseteq B_i \text{ and}$$
$$(A_1, A_2, A_3) \sim_i (B_1, B_2, B_3) :\Longleftrightarrow A_i = B_i \ (i = 1, 2, 3).$$

These quasiorders satisfy the *antiordinal dependences*: for $\{i, j, k\} = \{1, 2, 3\}$, $(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3)$ and $(A_1, A_2, A_3) \lesssim_j (B_1, B_2, B_3)$ imply $(A_1, A_2, A_3) \gtrsim_k (B_1, B_2, B_3)$ for all triconcepts (A_1, A_2, A_3) and (B_1, B_2, B_3) of $\mathbb{K}$. For $i \neq j$ the relation $\sim_i \cap \sim_j$ is the identity on $\mathfrak{T}(\mathbb{K})$, i.e. if we have two components of the triconcept the third one is uniquely determined by them. $[(A_1, A_2, A_3)]_i$ denotes the equivalence class of $\sim_i$ containing the triconcept (A_1, A_2, A_3). The quasiorder $\lesssim_i$ induces an order $\leq_i$ on the factor set $\mathfrak{T}(\mathbb{K})/\sim_i$ of all equivalence classes of $\sim_i$ which is given by $[(A_1, A_2, A_3)]_i \leq_i [(B_1, B_2, B_3)]_i \Leftrightarrow A_i \subseteq B_i$. $(\mathfrak{T}(\mathbb{K})/\sim_1, \leq_1), (\mathfrak{T}(\mathbb{K})/\sim_2, \leq_2)$ and $(\mathfrak{T}(\mathbb{K})/\sim_3, \leq_3)$ can be identified with the ordered set of all extents, intents and modi of $\mathbb{K}$ respectively. Generally every ordered set with smallest and greatest element is isomorphic to the ordered set of all extents, intents respectively modi of some triadic context. This means that the extents, intents respectively modi do not form a closure system in general as in the dyadic case.

An analogous structure to the concept lattice in the dyadic case is given by $\underline{\mathfrak{T}}(\mathbb{K}) := (\mathfrak{T}(\mathbb{K}), \lesssim_1, \lesssim_2, \lesssim_3)$. Let $\{i, j, k\} = \{1, 2, 3\}$, $\mathcal{X}_i$ and $\mathcal{X}_k$ be sets of triconcepts of $\mathbb{K}$ and $X_i := \bigcup\{A_i \mid (A_1, A_2, A_3) \in \mathcal{X}_i\}$ and $X_k := \bigcup\{A_k \mid (A_1, A_2, A_3) \in \mathcal{X}_k\}$. The *ik*-**join** of $(\mathcal{X}_i, \mathcal{X}_k)$ is defined to be the triconcept

$$\begin{aligned} \nabla(\mathcal{X}_i, \mathcal{X}_k) &:= (B_1, B_2, B_3) \text{ with} \\ B_i &:= X_i^{(i,j,X_k)(i,j,X_k)}, \\ B_j &:= X_i^{(i,j,X_k)}, \\ B_k &:= (X_i^{(i,j,X_k)(i,j,X_k)} \times X_i^{(i,j,X_k)})^{(k)}. \end{aligned}$$

The *Basic Theorem of triconcept Analysis* [7] proves that $\underline{\mathfrak{T}}(\mathbb{K})$ of a triadic context $\mathbb{K}$ is a complete trilattice with its *ik*-joins being exactly the *ik*-joins obtained by using the derivation operators. Further, every complete trilattice $\underline{L} := (L, \lesssim_1, \lesssim_2, \lesssim_3)$ is isomorphic to a concept trilattice of a suitable triadic context for which one can choose $(L, L, L, Y_{\underline{L}})$ with $Y_{\underline{L}} := \{(x_1, x_2, x_3) \in L^3 \mid$ there exists an $u \in L$ with $u \lesssim_i x_i$ for $i = 1, 2, 3\}$.

3 Factor Analysis through Formal Concept Analysis

This section contains the main results from [1] and the translation of the problem to Formal Concept Analysis. In [1] an approach to Factor Analysis is presented: a $p \times q$ binary matrix W is decomposed into the Boolean matrix product $P \circ Q$ of a $p \times n$ binary matrix P and a $n \times q$ binary matrix Q with n as small as possible. The Boolean matrix product $P \circ Q$ is defined as $(P \circ Q)_{ij} = \bigvee_{l=1}^{n} P_{il} \cdot Q_{lj}$, where $\bigvee$ is the maximum and $\cdot$ is the product. Through the Boolean matrix product a non-linear relationship between objects, factors and attributes is given.

The matrices W, P and Q represent an object-attribute, object-factor and factor-attribute relationship respectively. $W = P \circ Q$ means that object i is incident with attribute j if and only if there is a factor l which applies to i and contains j.

The method developed in [1] uses formal concepts as factors which produce decompositions with smallest number of factors possible. Contexts can be seen as Boolean matrices by replacing in the cross table the crosses by 1's and the blanks by 0's.

In the language of Formal Concept Analysis we give the following definition for factors:

Definition 4. *A subset of formal concepts $\mathcal{F} \subseteq \mathfrak{B}(G,M,I)$ such that $I = \bigcup_{(A,B)\in\mathcal{F}} A \times B$ is called* ***factorization****. If $\mathcal{F}$ is minimal with respect to its cardinality then it is called* ***optimal factorization****. The elements of $\mathcal{F}$ are called* ***(optimal) factors****.*

For subset $\mathcal{F} = \{(A_1,B_1),\ldots,(A_n,B_n)\} \subseteq \mathfrak{B}(G,M,I)$ of formal concepts we construct binary matrices $A_\mathcal{F}$ and $B_\mathcal{F}$ as follows:

$$(A_\mathcal{F})_{il} = \begin{cases} 1 \text{ if } i \in A_l \\ 0 \text{ if } i \notin A_l \end{cases}, \qquad (B_\mathcal{F})_{lj} = \begin{cases} 1 \text{ if } j \in B_l \\ 0 \text{ if } j \notin B_l \end{cases}$$

We consider a decomposition of the Boolean matrix W associated to (G,M,I) into the Boolean matrix product $A_\mathcal{F} \circ B_\mathcal{F}$.

Theorem 1. ***Universality of concepts as factors [1]***
For every W there is $\mathcal{F} \subseteq \mathfrak{B}(G,M,I)$ such that $W = A_\mathcal{F} \circ B_\mathcal{F}$.

Theorem 2. ***Optimality of concepts as factors [1]***
Let $W = P \circ Q$ for $p \times n$ and $n \times q$ binary matrices P and Q. Then there exists a set $\mathcal{F} \subseteq \mathfrak{B}(G,M,I)$ of formal concepts of W with $|\mathcal{F}| \leq n$ such that for the $p \times |\mathcal{F}|$ and $|\mathcal{F}| \times q$ binary matrices $A_\mathcal{F}$ and $B_\mathcal{F}$ we have $W = A_\mathcal{F} \circ B_\mathcal{F}$.

The proofs are based on the fact that a binary matrix can be seen as a $\bigvee$-superposition of rectangles full of crosses. Each such rectangle in contained in a maximal rectangle. Every maximal rectangle full of crosses corresponds to a formal concept.

Theorem 3. ***Mandatory factors [1]***
If $W = A_\mathcal{F} \circ B_\mathcal{F}$ for some subset $\mathcal{F} \subseteq \mathfrak{B}(G,M,I)$ of formal concepts then $\mathcal{O}(G,M,I) \cap \mathcal{A}(G,M,I) \subseteq \mathcal{F}$, where $\mathcal{O}(G,M,I)$ and $\mathcal{A}(G,M,I)$ are the sets of object and attribute concepts respectively.

The proof is based on the fact that if one considers a formal concept $(A,B) \in \mathcal{O}(G,M,I) \cap \mathcal{A}(G,M,I)$, then $(A,B) = (\{g\}'', \{g\}') = (\{m\}', \{m\}'')$ for some $g \in G$ and $m \in M$. The formal concept (A,B) is the only one which contains the tuple (g,m).

Theorem 4. *[1, 8]*
The problem to find a decomposition $W = P \circ Q$ of an $p \times q$ binary matrix W into an $p \times n$ binary matrix P and a $n \times q$ binary matrix Q with n as small as possible is NP-hard and the corresponding decision problem is NP-complete.

4 Triadic Factor Analysis

In this section $\mathbb{K}$ is a triadic context.

Definition 5. *A subset $\mathcal{F} \subseteq \mathfrak{T}(\mathbb{K})$ such that*

$$Y = \bigcup_{(A,B,C)\in\mathcal{F}} (A \times B \times C)$$

is called a ***factorization*** *of the triadic context $\mathbb{K}$. If $\mathcal{F}$ is minimal with respect to its cardinality then it is called* ***optimal factorization****. The elements of $\mathcal{F}$ are called* ***(optimal) factors****.*

In the dyadic case we have worked with Boolean matrices. For the triadic case we need the notion of a **Boolean** $3d$**-matrix** (shortly $3d$**-matrix**) which is a rectangular box $\mathbb{B}_{p\times q\times r}$ such that $b_{ijk} \in \{0,1\}$ for all $i \in \{1,\dots,p\}, j \in \{1,\dots,q\}, k \in \{1,\dots,r\}$. For a $3d$-matrix $\mathbb{B}$ we write $\mathbb{B} = \mathbb{B}_1 \mid \dots \mid \mathbb{B}_r$ where the $\mathbb{B}_k$ with $k \in \{1,\dots,r\}$ are $p \times q$ binary matrices called **layers**.

For a triadic context (K_1, K_2, K_3, Y) with $|K_1| = p, |K_2| = q, |K_3| = r$ we obtain the corresponding $3d$-matrix by replacing in the three-dimensional cross table the crosses by 1's and the blanks by 0's.

In order to define the $3d$ *Boolean matrix product* for a factorization $\mathcal{F} = \{(A_1,B_1,C_1),\dots,(A_n,B_n,C_n)\}$ of a triadic context $\mathbb{K}$ we first have to build matrices $A_\mathcal{F}, B_\mathcal{F}, C_\mathcal{F}$: $A_\mathcal{F}$ is defined as in the dyadic case and

$$(B_\mathcal{F})_{jl} = \begin{cases} 1 \text{ if } j \in B_l \\ 0 \text{ if } j \notin B_l \end{cases}, \qquad (C_\mathcal{F})_{kl} = \begin{cases} 1 \text{ if } k \in C_l \\ 0 \text{ if } k \notin C_l \end{cases}$$

for all $l \in \{1,\dots,n\}$.

$A_\mathcal{F}, B_\mathcal{F}, C_\mathcal{F}$ represent relationships between objects and factors, attributes and factors, conditions and factors respectively, i.e $(A_\mathcal{F})_{il} = 1$ means that object i can be described through factor l, $(B_\mathcal{F})_{jl} = 1$ means that attribute j is contained in factor l, and $(C_\mathcal{F})_{kl} = 1$ means that the factor l exists under the condition k.

Definition 6. *For $p\times n, q\times n$ and $r\times n$ binary matrices P, Q and R the* ***Boolean*** $3d$***-matrix product*** *(shortly* $3d$***-product****) is defined as a ternary operation:*

$$(P \circ Q \circ R)_{ijk} := \bigvee_{l=1}^{n} P_{il} \cdot Q_{jl} \cdot R_{kl}$$

with $i \in \{1,\dots,p\}, j \in \{1,\dots,q\}$ and $k \in \{1,\dots,r\}$.

For the $3d$-product defined above we consider two equivalent expressions. The first one is given by:

$$(P * Q \circ R)_{ijk} := \bigvee_{l=1}^{n} (P * Q)_{(ij)l} \cdot R_{kl} \text{ with}$$
$$(P * Q)_{(ij)l} := P_{il} \cdot Q_{jl}$$

where $P * Q$ is a $p \times q \times n$ $3d$-matrix and its n layers result from the Boolean multiplication of the l-th column $P_{\_l}$ of P and the l-th column $Q_{\_l}$ of Q. If we replace P and Q by $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ respectively then a layer l with $l \in \{1, \ldots, n\}$ of the $3d$-matrix describes the relationship between the objects and attributes of the l-th factor. Furthermore we have to check which factors exist under which conditions. Therefore we take $R = C_{\mathcal{F}}$ which describes the relationship between conditions and factors. The value of $(P * Q \circ R)_{ijk}$ is the maximum over the product between $(P * Q)_{(ij)l}$ and R_{kl} for each $l \in \{1, \ldots, n\}$. Finally $P * Q \circ R$ corresponds to the $3d$-matrix we wanted to decompose.

This $3d$-product mimics the best the situation of the dyadic case, since $P * Q$ describes the relation between objects and attributes. Because in the triadic case these tuples belong to different layers, we have to check under which conditions the tuples exist.

Consider the triadic context $\mathbb{K} = (\{1, 2, 3\}, \{a, b, c\}, \{b_1, b_2\}, Y)$ given by the three-dimensional cross table in Table 1. and the corresponding optimal factor-

	b_1			b_2		
	a	b	c	a	b	c
1	×	×			×	
2	×	×			×	
3	×		×	×		×

Table 1. Example of triadic context

ization $\mathcal{F} = \{(\{1, 2\}, \{a, b\}, b_1), (3, \{a, c\}, \{b_1, b_2\}), (\{1, 2\}, b, \{b_1, b_2\})\}$. Then the $3d$-product is given by:

$$\begin{aligned} A_{\mathcal{F}} * B_{\mathcal{F}} \circ C_{\mathcal{F}} &= \begin{pmatrix} 1\,0\,1 \\ 1\,0\,1 \\ 0\,1\,0 \end{pmatrix} * \begin{pmatrix} 1\,1\,0 \\ 1\,0\,1 \\ 0\,1\,0 \end{pmatrix} \circ \begin{pmatrix} 1\,1\,1 \\ 0\,1\,1 \end{pmatrix} = \\ &= \left(\begin{array}{c:c:c} 1\,1\,0 & 0\,0\,0 & 0\,1\,0 \\ 1\,1\,0 & 0\,0\,0 & 0\,1\,0 \\ 0\,0\,0 & 1\,0\,1 & 0\,0\,0 \end{array}\right) \cdot \begin{pmatrix} 1\,1\,1 \\ 0\,1\,1 \end{pmatrix} = \\ &= \left(\begin{array}{c:c} 1\,1\,0 & 0\,0\,0 \\ 1\,1\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 \end{array}\right) \bigvee \left(\begin{array}{c:c} 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 \\ 1\,0\,1 & 1\,0\,1 \end{array}\right) \bigvee \left(\begin{array}{c:c} 0\,1\,0 & 0\,1\,0 \\ 0\,1\,0 & 0\,1\,0 \\ 0\,0\,0 & 0\,0\,0 \end{array}\right) \end{aligned}$$

Another Boolean $3d$-matrix product is given by:

$$\begin{aligned} (P \circ Q * R)_{ijk} &:= \bigvee_{l=1}^{n} P_{il} \cdot (Q * R)_{(jk)l} \text{ with} \\ (Q * R)_{(jk)l} &:= Q_{jl} \cdot R_{kl} \end{aligned}$$

This $3d$ matrix product is very similar to the above one. However the difference is in the interpretation. The operations $\circ, *$ and $\cdot$ are defined as above but carried out on different matrices. If we replace P, Q and R by $A_\mathcal{F}, B_\mathcal{F}$ and $C_\mathcal{F}$ respectively then the lth layer of $B_\mathcal{F} * C_\mathcal{F}$ describes the relationship between attributes and conditions for the l-th factors with $l \in \{1, \ldots, n\}$. $P_{\_l} \cdot (Q * R)_{(\_\_)l}$ describes the relation between the objects and pairs of attributes and conditions for the l-th factor.

For the triadic context from above we obtain the following $3d$-product:

$$\begin{aligned} A_\mathcal{F} \circ B_\mathcal{F} * C_\mathcal{F} &= \begin{pmatrix} 1\,0\,1 \\ 1\,0\,1 \\ 0\,1\,0 \end{pmatrix} \cdot \begin{pmatrix} 1\,0 & 1\,1 & 0\,0 \\ 1\,0 & 0\,0 & 1\,1 \\ 0\,0 & 1\,1 & 0\,0 \end{pmatrix} = \\ &= \begin{pmatrix} 1\,1\,0 & 0\,0\,0 \\ 1\,1\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 \end{pmatrix} \bigvee \begin{pmatrix} 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 \\ 1\,0\,1 & 1\,0\,1 \end{pmatrix} \bigvee \begin{pmatrix} 0\,1\,0 & 0\,1\,0 \\ 0\,1\,0 & 0\,1\,0 \\ 0\,0\,0 & 0\,0\,0 \end{pmatrix} \end{aligned}$$

We give the triadic versions of the theorems presented in [1] showing that an optimal factorization of a triadic context can be obtained by triconcepts. The proofs are also very similar to the dyadic case, the difference consists in considering a different matrix product.

Theorem 5. *For every* $3d$*-matrix* $\mathbb{B}$ *there is* $\mathcal{F} \subseteq \mathfrak{T}(\mathbb{K})$ *such that:*

$$\mathbb{B} = A_\mathcal{F} \circ B_\mathcal{F} \circ C_\mathcal{F}.$$

Proof. $\mathbb{B}_{ijk} = 1$ iff there is a triconcept $(A, B, C) \in \mathfrak{T}(\mathbb{K})$ such that $i \in A$, $j \in B$ and $k \in C$. If we choose for $\mathcal{F}$ the set of all triconcepts $\mathfrak{T}(\mathbb{K})$ then $(A_\mathcal{F} \circ B_\mathcal{F} \circ C_\mathcal{F})_{ijk} = 1$ iff there is l such that $(A_\mathcal{F})_{il} = 1, (B_\mathcal{F})_{jl} = 1$ and $(C_\mathcal{F})_{kl} = 1$ iff there is $(A_l, B_l, C_l) \in \mathfrak{T}(\mathbb{K})$ with $i \in A_l, j \in B_l$ and $k \in C_l$ iff $\mathbb{B}_{ijk} = 1$. □

Using triconcepts as factors we obtain a factorization with smallest number of factors possible as stated by the next theorem:

Theorem 6. *Let* $\mathbb{K} = (K_1, K_2, K_3, Y)$ *be a triadic context and* $\mathbb{B}$ *the corresponding* $3d$*-matrix such that* $\mathbb{B} = P \circ Q \circ R$ *for* $p \times n, q \times n$ *and* $r \times n$ *binary matrices* P, Q *and* R*. Then there exists a set* $\mathcal{F} \subseteq \mathfrak{T}(\mathbb{K})$ *of triconcepts with* $|\mathcal{F}| \leq n$ *such that for the* $p \times |\mathcal{F}|, q \times |\mathcal{F}|$ *and* $r \times |\mathcal{F}|$ *binary matrices* $A_\mathcal{F}, B_\mathcal{F}$ *and* $C_\mathcal{F}$ *we have* $\mathbb{B} = A_\mathcal{F} \circ B_\mathcal{F} \circ C_\mathcal{F}$*.*

Proof. Recall that triconcepts are maximal rectangular boxes full of 1's which are maximal w.r.t. the component wise set inclusion.

Through $\mathbb{B} = P \circ Q \circ R$ for $p \times n, q \times n$ and $r \times n$ binary matrices P, Q and R, $\mathbb{B}$ is written as a $\bigvee$-superposition, as a union of n rectangular boxes full of 1's, but these rectangular boxes do not have to be maximal. $\mathbb{B} = P_{i1} \cdot Q_{j1} \cdot R_{k1} \vee \cdots \vee P_{in} \cdot Q_{jn} \cdot R_{kn}$ with $i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}$ and $k \in \{1, \ldots, r\}$. $\mathbb{B}$ is the union of rectangular boxes $\mathbb{B}_l = P_{\_l} \circ Q_{\_l} \circ R_{\_l}$. $\mathbb{B} = P \circ Q \circ R$ for $p \times n, q \times n$

and $r \times n$ binary matrices P, Q and R with corresponding rectangular boxes $\mathbb{B}_1, \ldots, \mathbb{B}_n$. Each such rectangular box $\mathbb{B}_l$ with $l \in \{1, \ldots, n\}$ is contained in a maximal rectangular box $\tilde{\mathbb{B}}_l$ of $\mathbb{B}$, i.e. $(\mathbb{B}_l)_{ijk} \leq (\tilde{\mathbb{B}}_l)_{ijk}$ for all i, j, k.
Every maximal rectangular box $\tilde{\mathbb{B}}_l$ corresponds to a triconcept (A_l, B_l, C_l) in that $(\tilde{\mathbb{B}}_l)_{ijk} = 1$ iff $i \in A_l, j \in B_l$ and $k \in C_l$. Since

$$\mathbb{B}_{ijk} = \bigvee_{l=1}^{n} (\mathbb{B}_l)_{ijk} \leq \bigvee_{l=1}^{n} (\tilde{\mathbb{B}}_l)_{ijk} \leq \mathbb{B}_{ijk}$$

a $\bigvee$-superposition of maximal rectangular boxes $\tilde{\mathbb{B}}_l$ yields $\mathbb{B}$. Putting therefore $\mathcal{F} = \{(A_1, B_1, C_1), \ldots, (A_n, B_n, C_n)\}$ we get $\mathbb{B} = A_{\mathcal{F}} \circ B_{\mathcal{F}} \circ C_{\mathcal{F}}$. Further $|\mathcal{F}| \lneqq n$ may happen because two distinct rectangular boxes can be contained in the same maximal rectangular box. □

The next definition contains the notion of *d-cut* which is actually a special case of $\mathbb{K}^{ij}_{X_k} = (K_i, K_j, Y^{ij}_{X_k})$ for $X_k \subseteq K_k$ and $|X_k| = 1$. Each d-cut is itself a dyadic context.

Definition 7. *For a triadic context $\mathbb{K} = (K_1, K_2, K_3, Y)$ a* ***dyadic-cut (d-cut)*** *is defined as:*

$$c^i_\alpha := (K_j, K_k, Y^{jk}_\alpha)$$

where $\{i, j, k\} = \{1, 2, 3\}$ and $\alpha \in K_i$.

For every triadic context there are three families of d-cuts:

$$c^1 := \{c^1_g := (K_2, K_3, Y^{23}_g)\}_{g \in K_1} \quad (1)$$
$$c^2 := \{c^2_m := (K_1, K_3, Y^{13}_m)\}_{m \in K_2} \quad (2)$$
$$c^3 := \{c^3_b := (K_1, K_2, Y^{12}_b)\}_{b \in K_3} \quad (3)$$

Hence, (3) represents horizontal cuts in $\mathbb{K}$ for each condition $b \in K_3$. The family $\{c^3_b\}_{b \in K_3}$ of d-cuts contains (at most) $|K_3|$ d-cuts. Such a d-cut is itself a dyadic context, namely (K_1, K_2, Y^{12}_b), and represents the relations between the object and attribute set of $\mathbb{K}$ under the condition $b \in K_3$. (1) represents vertical cuts in $\mathbb{K}$ for each object $g \in K_1$. Such a d-cut contains the relationship between all the attributes and conditions for the object which generated the d-cut. (2) represents also vertical cuts in $\mathbb{K}$ but this time one cuts for every attribute $m \in K_2$. Such a d-cut contains the relationship between all the objects and conditions for the attribute which generated the d-cut.

Obviously one can reconstruct the triadic context $\mathbb{K}$ from the d-cuts by 'gluing' them together. For the d-cut families of a triadic context the following equations hold:

$$Y = \bigcup_{g \in K_1} \{g\} \times Y^{23}_g = \bigcup_{m \in K_2} \{m\} \times Y^{13}_m = \bigcup_{b \in K_3} \{b\} \times Y^{12}_b.$$

Since each d-cut is a dyadic context, we can compute its concepts. For a d-cut $c^i_\alpha = (K_j, K_k, Y^{jk}_\alpha)$ its concepts are the pairs (A_j, A_k) such that for all $\{j, k\} \subseteq$

$\{1,2,3\}$ with $j < k$ and $A_j \subseteq K_j, A_k \subseteq K_k, \alpha \in A_i$ we have $A_j \mapsto A_j^{(j,k,\alpha)}$ and $A_k \mapsto A_k^{(j,k,\alpha)}$ with the derivation operators given in Section 3. From this concept of the d-cut we can compute the corresponding triconcept (A_1, A_2, A_3) with (A_j, A_k) given as above and $A_i \mapsto A_i^{(i)}$ for all $\{i,j,k\} = \{1,2,3\}$.

For different d-cuts of a d-cut family we may have identical dyadic contexts. This happens whenever the relationship between K_i and K_j are the same for some elements from K_k with $\{i,j,k\} = \{1,2,3\}$.

Definition 8. *A factor (A_1, A_2, A_3) is called **mandatory** if and only if there exists a tuple $(g,m,b) \in Y$ such that (A_1, A_2, A_3) is the only triconcept satisfying: $(g,m,b) \in A_1 \times A_2 \times A_3$.*

Theorem 7. *A factor (A_1, A_2, A_3) of a factorization $\mathbb{B} = A_{\mathcal{F}} \circ B_{\mathcal{F}} \circ C_{\mathcal{F}}$ for a subset $\mathcal{F} \subseteq \mathfrak{T}(\mathbb{K})$ of triconcepts is mandatory if and only if it is of the form*

$$\begin{aligned}
(A_1, A_2, A_3) &= (x_1^{(1,2,x_3)(1,2,x_3)}, x_1^{(1,2,x_3)}, (x_1^{(1,2,x_3)(1,2,x_3)} \times x_1^{(1,2,x_3)})^{(3)}) && (4)\\
&= (x_2^{(1,2,x_3)}, x_2^{(1,2,x_3)(1,2,x_3)}, (x_2^{(1,2,x_3)} \times x_2^{(1,2,x_3)(1,2,x_3)})^{(3)}) && (5)\\
&= (x_1^{(1,3,x_2)(1,3,x_2)}, (x_1^{1,3,x_2)(1,3,x_2)} \times x_1^{(1,2,x_3)})^{(2)}, x_1^{(1,3,x_2)}) && (6)\\
&= (x_3^{(1,3,x_2)}, (x_3^{(1,3,x_2)} \times x_3^{(1,2,x_3)(1,3,x_2)})^{(2)}, x_3^{(1,3,x_2)(1,3,x_2)}) && (7)\\
&= ((x_2^{(2,3,x_1)(2,3,x_1)} \times x_2^{(2,3,x_1)})^{(1)}, x_2^{(2,3,x_1)(2,3,x_1)}, x_2^{(2,3,x_1)}) && (8)\\
&= ((x_3^{(2,3,x_1)} \times x_3^{(2,3,x_1)(2,3,x_1)})^{(1)}, x_3^{(2,3,x_1)}, x_3^{(2,3,x_1)(2,3,x_1)}) && (9)
\end{aligned}$$

for fixed $x_\iota \in K_\iota$ with $\iota \in \{1,2,3\}$.

Remark 1. We know from the dyadic case that the mandatory factors are those which are object and attribute concepts simultaneously. In the triadic case we have for the d-cut family c^1 attribute and condition triconcepts $((4),(5))$, for c^2 object and condition triconcepts $((6),(7))$ and for c^3 object and attribute triconcepts $((8),(9))$. A factor is mandatory in the factorization of a triadic context if and only if it is a mandatory factor for at least one d-cut factorization of each d-cut family.

Proof. (A_1, A_2, A_3) as written in (4) and (5) is the only triconcept such that $x_\iota \in A_\iota$ with $\iota \in \{1,2,3\}$ in the d-cut $c^3_{x_3}$.

Suppose there is another triconcept $(B_1, B_2, B_3) \neq (A_1, A_2, A_3)$ from $c^3_{x_3}$. If $x_1 \in B_1$ then due to the properties of the triadic derivation operators we obtain: $A_1 = x_1^{(1,2,x_3)(1,2,x_3)} \subseteq B_1^{(1,2,x_3)(1,2,x_3)} = B_1$. Since (A_1, A_2, A_3) and (B_1, B_2, B_3) are triconcepts we have $A_2 = A_1^{(1,2,x_3)} \supseteq B_1^{(1,2,x_3)} = B_2$. On the other hand from $x_2 \in B_2$ follows $A_2 = x_2^{(1,2,x_3)(1,2,x_3)} \subseteq B_2^{(1,2,x_3)(1,2,x_3)} = B_2$. Altogether we obtain $B_2 = A_2$ and $A_1 = A_2^{(1,2,x_3)} = B_2^{(1,2,x_3)} = B_1$. Since two components of the triconcept uniquely determine the third one, we also have $B_3 = A_3$. This means that $(A_1, A_2, A_3) = (B_1, B_2, B_3)$. The triconcept (A_1, A_2, A_3) is the only one which contains the tuple (x_1, x_2, x_3) in the d-cut $c^3_{x_3}$.

Analogously we consider the cases (6), (7) and (8), (9). If a triconcept (A_1, A_2, A_3) can be written in the form of $(4) - (9)$ then is the only triconcept from $\mathfrak{T}(\mathbb{K})$ which contains the tuple (x_1, x_2, x_3). $\square$

A possibility to compute the factorization of a triadic context is through the d-cuts. One determines the optimal factorization of every d-cut of a d-cut family and takes their disjoint union as the factorization of the triadic context. However this method is naive and laborious since it may happen that we compute the same triconcept several times. Finally we also have to check the disjointnes of the dyadic factorizations. However Algorithm 1 eliminates this unnecessary verification.

The number of triconcepts even for a small triadic context can be quite big and searching for factors in $\mathfrak{T}(\mathbb{K})$ can get cumbersome. On the other hand the ik-joins obtained by triconcepts from different d-cuts usually increase the number of factors since they cover just partially the incidence of the ik-join components.

It follows trivially from the dyadic case that finding a factorization of a $3d$-matrix $\mathbb{B}$ into the $3d$ product $P \circ Q \circ R$ for $p \times n, q \times n$ and $r \times n$ binary matrices P, Q and R is NP-hard as well.

Algorithm 1 is the adoptions to the triadic case of the greedy approximation algorithm presented in [1]:[2]

Algorithm 1
Input: $\mathbb{B}$ (Boolean $3d$-matrix)
Output: $\mathcal{F}$ (set of factor concepts)

```
U := {(i, j, k) | B_{i,j,k} = 1}
for k := 1 to |K_3|
    U_k := {(i, j, k) | B_{i,j,k} = 1}
F := ∅
for k := 1 to |K_3|
    while U ≠ ∅ or U_k ≠ ∅:
      B := ∅
      V := 0
      while there is j ∉ B such that |B ⊕ j| ⪈ V:
         do select j ∉ B that maximizes B ⊕ j:
             B := (B ∪ j)^{(1,2,k)(1,2,k)}
             V := |(B^{(1,2,k)} × B) ∩ U|
      A := B^{(1,2,k)}
      C := (A × B)^{(3)}
      add (A, B, C) to F
      for each (i, j, k) ∈ A × B × C:
         remove (i, j, k) from U and from U_k
return F
```

[2] see also [3] for another algorithm and test data

Thereby $B \oplus j = ((B \cup j)^{(1,2,k)} \times (B \cup j)^{(1,2,k)(1,2,k)}) \cap U$. In this algorithm we consider the d-cut family of the conditions. Alternatively one can choose also another d-cut family. As presented in [1] for the dyadic case if (A, B) is a concept of a d-cut then the intent can be written as $B = \bigcup_{j \in B} j^{(1,2,k)(1,2,k)}$. Further if $j \notin B$ then $((B \cup \{j\})^{(1,2,k)}, (B \cup \{j\})^{(1,2,k)(1,2,k)})$ is a concept with $B \subset (B \cup \{j\})^{(1,2,k)(1,2,k)}$. This algorithm selects the triconcepts which cover as much of the incidence relation as possible. Finally we compute the third component of the triconcept. The sets U and U_k with $k \in \{1, \dots, |K_3|\}$ assure that we do not compute unnecessary triconcepts.

It may happen that for different d-cut families we obtain different factorization from which some are optimal and others not. It may happen that there exists a triconcept (A_1, A_2, A_3) in the factorization of a d-cut such that $A_1 \times A_2 \times A_3 \subseteq \bigcup_{(B_1,B_2,B_3) \in \mathcal{F} \setminus (A_1,A_2,A_3)} B_1 \times B_2 \times B_3$. An improvement of Algorithm 1 can be done by searching and removing such factors from the factorization.

The example we give is a small one and serves as an illustration of the methods developed in this paper. It is however a paradigmatic one since it reflects the usual setting in Web 2.0 applications. The object set contains the hostels {*Nuevo Suizo, Samay, Oasis Backpacker, One, Ole Backpacker, Garden Backpacker*}, the attribute set is given by the services {*character, safety, location, staff, fun, cleanliness*}. Since we have chosen the hostels with best ratings, the attribute values are ☻ (good) or ☺ (excellent) and are considered as tags. In the triadic context we make a cross in the corresponding line of object, attribute and condition if the service was rated as excellent. The conditions are given by the overall rating of users from { [9],[10],[11] }. We number consecutively the elements from K_1, K_2 and K_3, i.e $K_i := \{0, \dots, |K_i|\}$ with $i \in \{1, 2, 3\}$.

	0						1						2					
	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5
0			×						×	×					×	×		
1		×	×	×		×		×	×	×		×	×	×	×	×		×
2	×	×	×	×		×	×	×	×	×	×	×		×	×	×	×	×
3	×	×		×		×	×	×	×	×	×	×	×	×	×	×	×	×
4	×	×		×		×	×	×	×	×	×	×	×	×	×	×	×	×
5			×	×		×	×	×	×	×	×	×	×	×	×	×		×

Table 2. Triadic context of hostels

This triadic context has 18 triconcepts. To obtain the optimal factorization of the triadic context we consider the optimal factorizations of every d-cut from $\{c^3_b\}_{b \in K_3}$. The factorization of the d-cuts are given by $\mathcal{F}(c^1_1) = \{(\{0, 1, 2, 5\}, 2, K_3), (\{2, 3, 4\}, \{0, 1, 3, 5\}, \{0, 1\}), (\{1, 2, 5\}, \{2, 3, 5\}, K_3), (\{1, 2, 3, 4\}, \{1, 3, 5\}, K_3)\}$, $\mathcal{F}(c^1_2) = \{(K_1, \{2, 3\}, \{1, 2\}), (\{2, 3, 4, 5\}, K_2, 1), (\{1, 2, 3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2\})\}$ and $\mathcal{F}(c^1_3) = \{(K_1, \{2, 3\}, \{1, 2\}), (\{2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2\}), (\{1, 3, 4,$

$5\}, \{0,1,2,3,5\}, 2)\}$. Obviously $\mathcal{F}(c_2^1)$ and $\mathcal{F}(c_3^1)$ have a common triconcept. We can reduce the number of factors even more, since the incidence covered by the factor $(\{1,2,3,4,5\}, \{1,2,3,5\}, \{1,2\}) \in \mathcal{F}(c_2^1)$ is already covered by the factors $(K_1, \{2,3\}, \{1,2\}) \in \mathcal{F}(c_2^1)$ and $(\{1,2,3,4\}, \{1,3,5\}, K_3) \in \mathcal{F}(c_1^1)$. Algorithm 1 avoids including these 'unnecessary' triconcepts into the factorization. The factorization of the triadic context is given by $\mathcal{F} = \mathcal{F}(c_1^1)\dot{\cup}\mathcal{F}(c_2^1)\dot{\cup}\mathcal{F}(c_3^1) \setminus \{(\{1,2,3,4,5\}, \{1,2,3,5\}, \{1,2\})\}$.

The factors have also a verbal description, as for example the first one stand for *very safe* since the users from all platforms have this point of view, the hostels from the extent of $(\{2,3,4,5\}, K_2, 1)$ are the best deals according to the users from the second platform. On the other hand all users agreed that the hostels $1,2,5$ are excellent concerning safety, location and fun and that the hostels $1,2,3,4$ are excellent concerning character, location and fun.

The factorization offers the possibility to describe the hostels through 8 factors while in the triadic context they are described through 6 attributes under 3 conditions.

The $3d$-product can be written in a similar way like in the above examples. We omit the details due to lack of space.

5 Conclusion

We have presented a factorization method of triadic data such that the number of factors is as small as possible.

In this paper we have also illustrated the developed mathematical and algorithmic techniques through a paradigmatic example.

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