A First Peek into Preferential Logics with Team Semantics

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Abstract
This paper considers KLM-style preferential non-monotonic reasoning in the setting of propositional team semantics. We show that team-based propositional logics naturally give rise to cumulative non-monotonic entailment relations. Motivated by the non-classical interpretation of disjunction in team semantics, we give a precise characterization for preferential models for propositional dependence logic satisfying all of System P postulates. Furthermore, we show how classical entailment and dependence logic entailment can be expressed in terms of non-trivial preferential models.

Keywords
KLM, non-monotonic logic, System P, triangle-property, team logic, team semantics, preferential reasoning

1. Introduction

We define non-monotonic versions of team-based logics and study their axiomatics regarding System P. The logics are defined with the aid of preferential models in the style of Kraus, Lehmann and Magidor [1] (KLM).

Team semantics is a logical framework for studying concepts and phenomena that arise in the presence of plurality of data. Prime examples of such concepts are, e.g., functional dependence ubiquitous in database theory and conditional independence of random variables in statistics. The beginning of the field of team semantics can be traced back to the introduction of (first-order) dependence logic in [2]. In dependence logic, formulas are interpreted by sets of assignments (teams). Syntactically, dependence logic extends first-order logic by dependence atoms $=(\vec{x}, y)$ expressing that the values of the variables $\vec{x}$ functionally determine the value of the variable y. Inclusion logic [3] is another prominent logic in this context that extends first-order logic by inclusion atoms $\vec{x} \subseteq \vec{y}$, whose interpretation corresponds exactly to that of inclusion dependencies in database theory. During the past decade, the expressivity and complexity aspects of logics in team semantics have been extensively studied. Fascinating connections have been drawn to areas such as database theory [4, 5], verification [6], real-valued computation [7], inquisitive logic [8], and epistemic logic [9]. These works have focused on logics in the first-order, propositional and modal team semantics, and more recently also in the multiset [10], probabilistic [11] and semiring settings [12]. As far as the authors know, a merger of logics in team semantics and non-monotonic reasoning has not been studied so far.

Non-monotonicity is one of the core phenomenons of reasoning that are deeply studied in knowledge representation and reasoning; see Gabbay et al. (1993) and Brewka et al. (1997) for an overview, with, e.g., connections to belief change [15] and human-like reasoning [16]. Non-monotonic inference $\varphi \mathrel{|\!\sim} \psi$ is often understood as "when φ holds, then usually ψ holds", where usually can be understood in the sense of *expected* [17]. One can imagine adapting this notion of non-monotonic inference to propositional team logics. For instance, in dependence logic, an entailment $=(b, f) \models \neg p$ states that

"when whether it is a bird *(b) determines whether it* flies *(f), then it is not a penguin ($\neg p$)"*

and an analogue non-monotonic entailment $=(b, f) \mathrel{|\!\sim} \neg p$ can be read as

"when whether it is a bird *(b) determines whether it* flies *(f), then it is* ***usually*** *not a penguin ($\neg p$)".*

Alternatively to the interpretation above, one can understand non-monotonic inferences from a team perspective. For example, $=(b, f) \mathrel{|\!\sim} \neg p$ reads then as "a team that usually satisfies $=(b, f)$ also satisfies $\neg p$". For the latter kind of expression, there is no obvious way to formulate it in existing team-based logic, so injecting non-monotonicity is a valuable extension of team logics. Note that "$=(b, f) \mathrel{|\!\sim} \neg p$" does not imply that $=(b, f) \land p$ is inconsistent. The semantics of team logic is developed with emphasis on teams. Depended on the application context, one reads $=(b, f) \mathrel{|\!\sim} \neg p$, e.g., as follows:

Database Interpretation: "When the value of b determines the value of f in a database, then usually the value of p is 0."

Possible World Interpretation. "When the agent is convinced that whether f holds in a world always depends on b, then usually the agent believes that p does not hold."

There are several approaches to non-monotonic reasoning, e.g., circumscription, autoepistemic logic, Reiters default logic, see Gabbay (1993) for an overview. For a start, one can rely on the basic systems of non-monotonic reasoning. The very most basic denominator of non-monotonic reasoning is often denoted cumulative reasoning, which is given axiomatically by *System C* [18]. In extension to cumulative reasoning, non-monotonic reasoning in the style of KLM is considered as the "conservative core of non-monotonic reasoning" [19, 18]. KLM-style non-monotonic reasoning has two prominent representations [1]:

(KLM.1) reasoning over *preferential models*; and

(KLM.2) an axiomatic characterization, called *System P*, which is an extension of System C.

Because of (KLM.1), KLM-style reasoning is also denoted *preferential reasoning*. Common for both representations of KLM-style reasoning is, that they are parametric in the sense that they make use of some underlying classical logic $\mathscr{L}$, e.g., propositional logic or first-order logic.

In this paper, we define preferential team logics via preferential models (as in KLM.1). The rationale is that we think that preferential models capture the original intention of preferential logic best, and, as we demonstrate, it shows standard non-monotonic behaviour. Furthermore, we study

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the relationship of preferential teams logic to System P (as in KLM.2). Our axiomatic studies show that for general team-based logics, (KLM.1) and (KLM.2) do not induce the same non-monotonic inference relations. This is of interest, e.g., because it gives a negative answer to the question of whether the relationship between (KLM.1) and (KLM.2) by KLM (1990) generalize beyond the assumptions by KLM[1]. We give a condition for preferential models that is sufficient to reestablish satisfaction of System P in all preferential team logics. Specifically for preferential dependence logic, we also show that this condition exactly characterizes those preferential models such that System P is satisfied. Moreover, when using specific (non-trivial) preferences, preferential dependence logic becomes dependence logic, respectively, it is equivalent to classical propositional entailment.

2. Background: Team-Based Logics

In this section we present the background on propositional logics with team semantics, propositional dependence logic and propositional inclusion logic (see [20] for a survey on team-based logics).

2.1. Propositional Logic with Team Semantics

We denote by $\mathsf{Prop} = \{p_i \,:\, i \in \mathbb{N}\}$ the set of propositional variables. We will use letters $p, q, r, \ldots$ (with or without subscripts) to stand for elements of Prop. In this article, we consider only formulas in negation normal form.

Definition 1 (Classical propositional logic (PL))**.** Well formed PL-formulas α are formed by the grammar:

$$\alpha ::= p \mid \neg p \mid \bot \mid \top \mid \alpha \wedge \alpha \mid \alpha \vee \alpha$$

In team semantics, one usually considers a non-empty finite subset $N \subseteq \mathsf{Prop}$ of propositional variables and defined for valuations $v : N \to \{0,1\}$ over N and PL-formulas α:

$$[\![\alpha]\!]^c := \{v : N \to \{0,1\} \mid v \models \alpha\}.$$

We write $v \models p$ in case $v(p) = 1$, and $v \not\models p$ otherwise. The valuation function v is extended to the set of all PL-formulas in the usual way.

Definition 2. For any set $\Delta \cup \{\alpha\}$ of PL-formulas, we write $\Delta \models^c \alpha$ if for any valuation v, $v \models \delta$ for all $\delta \in \Delta$ implies $v \models \alpha$. We write simply $\alpha \models^c \beta$ for $\{\alpha\} \models^c \beta$ and $\alpha \equiv^c \beta$ if both $\alpha \models^c \beta$ and $\beta \models^c \alpha$.

Next we define *team semantics* for PL-formulas (cf. [21, 22]). A team X is a set of valuations for some finite set $N \subseteq \mathsf{Prop}$. We write $\mathsf{dom}(X)$ for the domain N of X.

Definition 3 (Team semantics of PL)**.** Let X be a team. For any PL-formula α with $\mathsf{dom}(X) \supseteq \mathsf{Prop}(\alpha)$, the satisfaction relation $X \models \alpha$ is defined inductively as:

- $X \models p$ iff for all $v \in X$, $v \models p$;
- $X \models \neg p$ iff for all $v \in X$, $v \not\models p$;
- $X \models \bot$ iff $X = \emptyset$;
- $X \models \top$ is always the case;
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$;
- $X \models \alpha \vee \beta$ iff there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \alpha$ and $Z \models \beta$.

The set of all teams X with $X \models \alpha$ is written as $[\![\alpha]\!]$. Logical entailment and equivalence are defined as usual. For any set $\Delta \cup \alpha$ of classical formulas, we write $\Delta \models \alpha$ if for any team X, $X \models \delta$ for all $\delta \in \Delta$ implies $X \models \alpha$. We write simply $\alpha \models \beta$ for $\{\alpha\} \models \beta$. Write $\alpha \equiv \beta$ if both $\alpha \models \beta$ and $\beta \models \alpha$.

Proposition 4. *Let α be a PL-formula. Then the following properties hold:*

Flatness: $X \models \alpha \iff$ *for all* $v \in X$, $\{v\} \models \alpha$.

Empty team property: $\emptyset \models \alpha$.

Downwards closure: *If* $X \models \alpha$ *and* $Y \subseteq X$, *then* $Y \models \alpha$.

Union closure: *If* $X \models \alpha$ *and* $Y \models \alpha$, *then* $X \cup Y \models \alpha$.

For any PL-formula α, it further holds that

$$\{v\} \models \alpha \iff v \models \alpha,$$

and hence for classical formulas, $\Delta \models^c \alpha \iff \Delta \models \alpha$.

2.2. Propositional Dependence and Inclusion Logic

A *(propositional) dependence atom* is a string $=\!(a_1 \ldots a_k, b)$, and a *(propositional) inclusion atom* is a string $a_1 \ldots a_k \subseteq b_1 \ldots b_k$, in which $a_1, \ldots, a_k, b, b_1, \ldots, b_k$ are propositional variables from Prop. The team semantics of these two types of atoms is defined as follows, whereby $\vec{a}$ stands for $a_1, \ldots, a_k$:

- $X \models =\!(\vec{a}, b)$ iff for all $v, v' \in X$, $v(\vec{a}) = v'(\vec{a})$ implies $v(b) = v'(b)$.
- $X \models \vec{a} \subseteq \vec{b}$ iff for all $v \in X$, there exists $v' \in X$ such that $v(\vec{a}) = v'(\vec{b})$.

We define *propositional dependence logic* (denoted as $\mathsf{PL}(=\!(,))$) as the extension of PL-formulas with dependence atoms. Similarly, *propositional inclusion logic* (denoted as $\mathsf{PL}(\subseteq)$) is the extension of PL by inclusion atoms. In this paper, we use *propositional team logic* to refer to any of the logics PL, $\mathsf{PL}(=\!(,))$ and $\mathsf{PL}(\subseteq)$.

It is straightforward to check that dependence atoms do not have the union closure property and inclusion atoms the downwards closure property. However, the following holds.

Proposition 5. *Formulas of* $\mathsf{PL}(=\!(,))$ *and* $\mathsf{PL}(\subseteq)$ *have the empty team property. Moreover,* $\mathsf{PL}(=\!(,))$*-formulas have the downwards closure property, while* $\mathsf{PL}(\subseteq)$*-formulas have the union closure property.*

A dependence atom with the empty sequence in the first component will be abbreviated as $=\!(p)$ and called *constancy atoms*. The team semantics of constancy atoms is reduced to

- $X \models =\!(p)$ iff for all $v, v' \in X$, $v(p) = v'(p)$.

Example 6. Consider the team X over $\{p, q, r\}$ defined by:

[1] KLM assume a compact Tarskian logic with Boolean connectives. In team logics (by default), there is no negation, and disjunction is non-classical, i.e., it does not behave like Boolean disjunction.

	p	q	r
v_1	1	0	0
v_2	0	1	0
v_3	0	1	0

We have $X \models {=}(p,q)$ and $X \models {=}(r)$. Moreover, $X \models {=}(p) \vee {=}(p)$ but $X \not\models {=}(p)$. It is worth noting that $\mathsf{PL}(\subseteq)$-formulas ϕ satisfy

$$\phi \equiv \phi \vee \phi$$

because of the union closure property.

We can define the flattening ϕ^f of a $\mathsf{PL}(=(,))$-formula by replacing all dependence atoms by $\top$. It is easy to check that $\phi \models \phi^f$ and that

$$\{s\} \models \phi \Leftrightarrow s \models \phi^f \tag{1}$$

for all assignments s using the fact that dependence atoms are always satisfied by singletons.

3. Background: Preferential Logics

In this section, we present background on preferential logics in style of Kraus, Lehmann and Magidor [1].

3.1. Preferential Models and Entailment

In preferential logic, an entailment $\varphi \mid\!\sim \psi$ holds, when minimal models of φ are models of ψ. This is formalized via preferential models, which we introduce in the following.

For a strict partial order $\prec\ \subseteq \mathcal{S} \times \mathcal{S}$ on a set $\mathcal{S}$ and a subset $S \subseteq \mathcal{S}$, an element $s \in S$ is called *minimal in S with respect to $\prec$* if for each $s' \in S$ holds $s' \not\prec s$. Then, $\min(S, \prec)$ is the set of all $s \in S$ that are minimal in S with respect to $\prec$.

Definition 7 ([1])**.** Let $\mathscr{L}$ be a logic and Ω be the set of interpretations for $\mathscr{L}$. A *preferential model* for $\mathscr{L}$ is a triple $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ where $\mathcal{S}$ is a set, $\ell : \mathcal{S} \to \Omega$, $\prec$ is a strict partial order on $\mathcal{S}$, and the following condition is satisfied:

> *[Smoothness]* $S(\varphi) = \{s \in \mathcal{S} \mid \ell(s) \models \varphi\}$ is smooth with respect to $\prec$ for every formula $\varphi \in \mathscr{L}$, i.e, for each $s \in S(\varphi)$ holds
> – s is minimal in $S(\varphi)$ with respect to $\prec$ or
> – there exists an $s' \in S(\varphi)$ that is minimal in $S(\varphi)$ with respect to $\prec$ with $s' \prec s$.

Smoothness guarantees the existence of minimal elements.

Definition 8 ([1])**.** The entailment relation $\mid\!\sim_{\mathbb{W}} \subseteq \mathscr{L} \times \mathscr{L}$ for a preferential model $\mathbb{W}$ over a logic $\mathscr{L}$ is given by

$$\varphi \mid\!\sim_{\mathbb{W}} \psi \text{ if for all } s \in \min(S(\varphi), \prec) \text{ holds } \ell(s) \models \psi$$

An entailment relation $\mid\!\sim\ \subseteq \mathscr{L} \times \mathscr{L}$ is called *preferential* if there is a preferential model $\mathbb{W}$ for $\mathscr{L}$ such that $\mid\!\sim\ =\ \mid\!\sim_{\mathbb{W}}$.

Because there are many preferential models for a logic $\mathscr{L}$, we may have for one logic $\mathscr{L}$ with multiple preferential logics that are based on $\mathscr{L}$. More precisely, when one considers a logical language $\mathcal{L}$, an entailment relation $\models$ over $\mathcal{L}$ that is based on a model theory with interpretations Ω, then there are (infinitely) many different preferential models $\mathbb{W}_1, \mathbb{W}_2, \ldots$ for this logic. Many of these preferential models yield different entailment relations $\mid\!\sim_{\mathbb{W}_1}, \mid\!\sim_{\mathbb{W}_2}, \ldots$.

3.2. Axiomatic Characterization by System P

We make use of the following rules for non-monotonic entailment $\mid\!\sim$:

$$\frac{}{\alpha \mid\!\sim \alpha}\ \text{(Ref)} \qquad \frac{\alpha \models \beta \quad \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}\ \text{(RW)}$$

$$\frac{\alpha \equiv \beta \quad \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}\ \text{(LLE)} \qquad \frac{\alpha \mid\!\sim \beta \quad \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}\ \text{(CM)}$$

$$\frac{\alpha \wedge \beta \mid\!\sim \gamma \quad \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}\ \text{(Cut)} \qquad \frac{\alpha \mid\!\sim \gamma \quad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}\ \text{(Or)}$$

Note that $\models$ is the entailment relation of the underlying monotonic logic $\mathscr{L}$. The rules (Ref), (RW), (LLE), (CM) and (Cut) forming *System C*. The rule (CM) goes back to the foundational paper on non-monotonic reasoning system by Gabbay (1984) and is a basic wakening of monotonicity. *System P* consists of all rules of System C and the rule (Or). The rule of (Or) is motivated by reasoning by case [19]. KLM showed a direct correspondence between preferential entailment relations and entailment relations that satisfy System P.

Proposition 9 (Kraus et al. 1990)**.** *Let $\mathscr{L}$ be a compact Tarskian logic with all Boolean connectives. A entailment relation $\mid\!\sim\ \subseteq \mathscr{L} \times \mathscr{L}$ satisfies System P if and only if $\mid\!\sim$ is preferential.*

4. Preferential Team Logics

For propositional team-based logics, we restrict ourselves to preferential models that we call standard.

Definition 10. A preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ is called standard if

(S1) There is no state $s \in \mathcal{S}$ such that $\ell(s) = \emptyset$

(S2) For all non-empty teams X there is some state $s \in \mathcal{S}$ such that $\ell(s) = X$

The rationale for (S1) and (S2) is to make the models concise and meaningful, i.e., containing explicit, yet necessary information for specifying reasoning. By (S1) we are excluding the empty team $\emptyset$ from $\mathcal{S}$, because team logics considered here have the empty-team property. Hence, $\emptyset$ is trivially a model of every formula and including it provides no extra information. Condition (S2) ensures that every "non-trivial" model is included, and thus, its preference status is explicitly given in the preferential model.

We define the family of preferential team logics as those that are induced by some standard preferential model.

Definition 11. A entailment relation $\mid\!\sim$ over some propositional team logic is called (standard) preferential, if there is some standard preferential model $\mathbb{W}$ such that $\mid\!\sim\ =\ \mid\!\sim_{\mathbb{W}}$.

The next example is the bird-penguin example, demonstrating that preferential team logics are indeed non-monotonic.

Example 12. Fix the set of propositional variables $N = \{b, p, f\} \subseteq \mathsf{Prop}$, with the following intended meanings: b stands for *"it is a bird"*, p stands for *"it is a penguin"*, and

f stands for *"it is able to fly"*. We construct a (standard) preferential model, by using the following teams:

$$X_{b\overline{p}f} = \begin{array}{c|ccc} & b & p & f \\ \hline v_1 & 1 & 0 & 1 \end{array} \quad X_{bp\overline{f}} = \begin{array}{c|ccc} & b & p & f \\ \hline v_2 & 1 & 1 & 0 \end{array}$$

Let $\mathbb{W}_{\text{peng}} = \langle \mathcal{S}_{\text{peng}}, \ell_{\text{peng}}, \prec_{\text{peng}} \rangle$ be the preferential model such that $\mathcal{S}_{\text{peng}} = \{s_X \mid X \text{ is a non-empty team}\}$ and $\ell_{\text{peng}}(s_X) = X$; for all singleton teams X different from $X_{b\overline{p}f}$ and $X_{bp\overline{f}}$ we define:

$$X_{b\overline{p}f} \prec_{\text{peng}} X_{bp\overline{f}} \prec_{\text{peng}} X$$

for all non-empty teams Y and non-empty non-singleton teams Z we define:

$$Y \prec_{\text{peng}} Z \text{ if } Y \subsetneq Z$$

Then, for $\mathrel{|\!\sim} = \mathrel{|\!\sim}_{\mathbb{W}_{\text{peng}}}$ we obtain the following inference:

$$b \mathrel{|\!\sim} f \quad \text{("birds usually fly")}$$
$$p \mathrel{|\!\sim} \neg f \quad \text{("penguins usually do not fly")}$$
$$b \wedge p \mathrel{|\!\not\sim} f \quad \text{("penguin birds usually do not fly")}$$

This is because we have:

$$\min(\llbracket b \rrbracket, \prec_{\text{peng}}) = \{X_{b\overline{p}f}\} \subseteq \llbracket f \rrbracket$$
$$\min(\llbracket p \rrbracket, \prec_{\text{peng}}) = \min(\llbracket b \wedge p \rrbracket, \prec_{\text{peng}}) = \{X_{bp\overline{f}}\} \subseteq \llbracket \neg f \rrbracket$$

Note that Example 12 is agnostic about the concrete team logic used, i.e., it applies to PL, $\mathsf{PL}(=(,))$, and $\mathsf{PL}(\subseteq)$.

5. General Axiomatic Evaluation

We will now present general results on whether System P holds for non-preferential and preferential team logics.

5.1. System P and Non-Preferential Team Logics

For the entailment $\models$ of propositional team logics, we obtain that System P is not satisfied by $\mathsf{PL}(=(,))$. For PL and $\mathsf{PL}(\subseteq)$, we obtain that they satisfy System P.

Proposition 13. *The following statements hold for* $\models$*:*

(a) $\mathsf{PL}(=(,))$ *satisfies System C, but violates System P.*

(b) PL *(under team semantics) and* $\mathsf{PL}(\subseteq)$ *satisfy System P.*

Proof. We show both statements.

(a) Satisfaction of System C is a corollary of Proposition 14 and (b) of Proposition 23. The violation of (Or) is witnessed by choosing α, β and γ to be the formula $=(p)$ in Example 6.

(b) We start with satisfaction of System C. Note that one can reconstruct non-preferential entailment $\models$ of PL by using a preferential model where all teams are incomparable. In such a preferential model $\mathbb{W}$ one has $\min(\llbracket\alpha\rrbracket, \prec) = \llbracket\alpha\rrbracket$. Hence, we have $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \beta$ if and only if $\llbracket\alpha\rrbracket \subseteq \llbracket\beta\rrbracket$ if and only if $\alpha \models \beta$. By using this, satisfaction of System C is a corollary of Proposition 14.
It remains to show that (Or) is satisfied. Let A, B and C be PL-formulas such that $A \models C$ and $B \models C$. If X is a model of $A \vee B$, then there are teams Y, Z with $X = Y \cup Z$ such that $Y \models A$ and $Z \models B$. Because Y, Z are models of C and because PL has the union closure property (see Proposition 5), we obtain that X is also a model of C. Hence, $A \vee B \models C$. The proof of statement (b) for $\mathsf{PL}(\subseteq)$ is the same. ☐

Note that Example 6 is a witness for the second part of the statement (a) of Proposition 13, i.e., $\mathsf{PL}(=(,))$ violates (Or).

5.2. System P and Preferential Team Logics

Generally, System C is satisfied by preferential team logics.

Proposition 14. *Let* $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ *be a preferential model for a propositional team logic. The preferential entailment relation* $\mathrel{|\!\sim}_{\mathbb{W}}$ *satisfies System C.*

Proof. We show that $\mathrel{|\!\sim}_{\mathbb{W}}$ satisfies all rules of System C, i.e., Ref, LLE, RW, Cut, and CM.

[*Ref.*] Considering the definition of $\mathrel{|\!\sim}_{\mathbb{W}}$ yields that $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \alpha$ if for all minimal $s \in S(\alpha)$ holds $\ell(s) \models \alpha$. By the definition of $S(\alpha)$, we have $s \in S(\alpha)$ if $\ell(s) \models \alpha$. Consequently, we have $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \alpha$.

[*LLE.*] Suppose that $\alpha \equiv \beta$ and $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \gamma$ holds. From $\alpha \equiv \beta$, we obtain that $S(\alpha) = S(\beta)$ holds. By using this last observation and the definition of $\mathrel{|\!\sim}_{\mathbb{W}}$, we obtain $\beta \mathrel{|\!\sim}_{\mathbb{W}} \gamma$ from $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \gamma$.

[*RW.*] Suppose that $\alpha \models \beta$ and $\gamma \mathrel{|\!\sim}_{\mathbb{W}} \alpha$ holds. Clearly, by definition of $\alpha \models \beta$ we have $\llbracket\alpha\rrbracket \subseteq \llbracket\beta\rrbracket$. From the definition of $\gamma \mathrel{|\!\sim}_{\mathbb{W}} \alpha$, we obtain that $\ell(s) \models \alpha$ holds for each minimal $s \in S(\gamma)$. The condition $\ell(s) \models \alpha$ in the last statement is equivalent to stating $\ell(s) \in \llbracket\alpha\rrbracket$. Because of $\llbracket\alpha\rrbracket \subseteq \llbracket\beta\rrbracket$, we also have $\ell(s) \in \llbracket\beta\rrbracket$; and hence, $\ell(s) \models \beta$ for each minimal $s \in S(\gamma)$. This shows that $\gamma \mathrel{|\!\sim}_{\mathbb{W}} \beta$ holds.

[*Cut.*] Suppose that $\alpha \wedge \beta \mathrel{|\!\sim}_{\mathbb{W}} \gamma$ and $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \beta$ holds. By unfolding the definition of $\mathrel{|\!\sim}_{\mathbb{W}}$, we obtain $\min(S(\alpha \wedge \beta), \prec) \subseteq S(\gamma)$ from $\alpha \wedge \beta \mathrel{|\!\sim}_{\mathbb{W}} \gamma$. Analogously, $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \beta$ unfolds to $\min(S(\alpha), \prec) \subseteq S(\beta)$. Moreover, employing basic set theory yields that $S(\alpha \wedge \beta) = S(\alpha) \cap S(\beta) \subseteq S(\alpha)$ holds. From $S(\alpha \wedge \beta) \subseteq S(\alpha)$ and $\min(S(\alpha), \prec) \subseteq S(\beta)$, we obtain $\min(S(\alpha), \prec) \subseteq S(\alpha \wedge \beta)$. Consequently, we also have that $\min(S(\alpha), \prec) = \min(S(\alpha \wedge \beta), \prec)$ holds. Using the last observation and $\min(S(\alpha \wedge \beta), \prec) \subseteq S(\gamma)$, we obtain $\min(S(\alpha), \prec) \subseteq S(\gamma)$. Hence also $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \gamma$ holds.

[*CM.*] Suppose that $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \beta$ and $\alpha \mathrel{|\!\sim}_{\mathbb{W}} \gamma$ holds. By unfolding the definition of $\mathrel{|\!\sim}_{\mathbb{W}}$, we obtain $\min(S(\alpha), \prec) \subseteq S(\beta)$ and $\min(S(\alpha), \prec) \subseteq S(\gamma)$. We have to show that $\min(S(\alpha \wedge \beta), \prec) \subseteq S(\gamma)$ holds. Let s be element of $\min(S(\alpha \wedge \beta), \prec)$. Clearly, we have that $s \in S(\alpha)$ holds. We show by contradiction that s is minimal in $S(\alpha)$. Assume that s is not minimal in $S(\alpha)$. From the smoothness condition, we obtain that there is an $s' \in S(\alpha)$ such that $s' \prec s$ and s' is minimal in $S(\alpha)$ with respect to $\prec$. Because s' is minimal and because we have $\min(S(\alpha), \prec) \subseteq S(\beta)$, we also have that $s' \in S(\beta)$ holds and hence that $s' \in S(\alpha \wedge \beta)$ holds. The latter contradicts the minimality of s in $S(\alpha \wedge \beta)$. Consequently, we have that $s \in \min(S(\alpha), \prec)$ holds. Because we have $\min(S(\alpha), \prec) \subseteq S(\gamma)$, we obtain $\alpha \wedge \beta \mathrel{|\!\sim}_{\mathbb{W}} \gamma$. □

The following Example 15 witnesses that, in general, (Or), and hence, System P, is violated by preferential team logics.

Example 15. Assume that $N = \{p, q\} \subseteq \mathsf{Prop}$ holds. The following valuations v_1, v_2, v_3 will be important:

$$v_1(p) = v_1(q) = v_2(q) = 1 \quad v_2(p) = v_3(p) = v_3(q) = 0$$

We consider the teams $X_{pq} = \{v_1\}$, $X_{\overline{p}q} = \{v_2\}$, and $X_{p\leftrightarrow q} = \{v_1, v_3\}$. Let $\mathbb{W}_{\mathrm{pq}} = \langle \mathcal{S}_{\mathrm{pq}}, \ell_{\mathrm{pq}}, \prec_{\mathrm{pq}} \rangle$ be the preferential model such that

$$\mathcal{S}_{\mathrm{pq}} = \{s_X \mid X \text{ is a non-empty team}\} \quad \ell_{\mathrm{pq}}(s_X) = X$$

holds, and such that $\prec_{\mathrm{pq}}$ is the strict partial order given by[2]

$$\begin{array}{ll} X_{p\leftrightarrow q} \prec_{\mathrm{pq}} X_{pq} & X_{pq} \prec_{\mathrm{pq}} X \\ X_{p\leftrightarrow q} \prec_{\mathrm{pq}} X_{\overline{p}q} & X_{\overline{p}q} \prec_{\mathrm{pq}} X \end{array}$$

where X stands for every team different from $X_{\overline{p}q}$ and $X_{p\leftrightarrow q}$. We obtain the following preferential entailments:

$$p \mathrel{|\!\sim}_{\mathbb{W}_{\mathrm{pq}}} q \qquad \neg p \mathrel{|\!\sim}_{\mathbb{W}_{\mathrm{pq}}} q \qquad p \vee \neg p \mathrel{|\!\not\sim}_{\mathbb{W}_{\mathrm{pq}}} q$$

Proposition 16. *The entailment relation $\mathrel{|\!\sim}_{\mathbb{W}_{pq}}$ for PL, respectively $\mathsf{PL}(=\!(,))$ and $\mathsf{PL}(\subseteq)$, violates (Or).*

We can reestablish satisfaction of System P, by demanding the ($\star$)-property, which we define below in Proposition 17. In the following we abuse notation and mean by $\min([\![A]\!], \prec)$ the set of $\prec$-minimal states in $\mathcal{S}(A)$, as well as the set of al models $\ell(s)$ of A for which a $\prec$-minimal states s in $\mathcal{S}(A)$ exists. More technically correct would be to write $\min(S(A), \prec)$ for the former, and writing $\{\ell(s) \mid s \in \min(S(A), \prec)\}$ for the latter.

Proposition 17. *Let $\mathbb{W}$ be a preferential model for some preferential team logic. If ($\star$) is satisfied for all formulas A, B, then $\mathrel{|\!\sim}_{\mathbb{W}}$ satisfies System P, whereby[3]:*

$$\min([\![A \vee B]\!], \prec) \subseteq \min([\![A]\!], \prec) \cup \min([\![B]\!], \prec) \qquad (\star)$$

Proof. Suppose that $A \mathrel{|\!\sim} \gamma$ and $B \mathrel{|\!\sim} \gamma$ holds. This is the same as $\min([\![A]\!], \preceq) \subseteq [\![\gamma]\!]$ and $\min([\![B]\!], \preceq) \subseteq [\![\gamma]\!]$. Because ($\star$) holds, this also means that $\min([\![A \vee B]\!], \preceq) \subseteq [\![\gamma]\!]$ holds. □

6. Results for Preferential Dependence Logics

For preferential dependence logic, we provide additional results to those of Section 5.

6.1. System P and Preferential Dependence Logic

The main contribution is a characterization of exactly those preferential entailment relations that satisfy all rules of System P. Central to this result is the following property for a preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$, where $s, s' \in \mathcal{S}$ are states:

$$\text{for all } s,\ |\ell(s)| > 1, \text{exists } s' \text{ with } \ell(s') \subsetneq \ell(s) \text{ and } s' \prec s \qquad (\triangle)$$

The ($\triangle$)-property demands (when understanding states as teams) that for each non-singleton team X exists a proper subteam Y of X that is preferred over X. For this property, we can show the following theorem.

Theorem 18. *Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ be a preferential model for $\mathsf{PL}(=\!(,))$. The following statements are equivalent:*

(i) $\mathrel{|\!\sim}_{\mathbb{W}}$ satisfies System P.

(ii) $\mathbb{W}$ satisfies the $\triangle$-property.

(iii) The ($\star$)-property holds for all $A, B \in \mathsf{PL}(=\!(,))$.

We will obtain the proof of the theorem via the following lemmata.

For the first lemma, assume that $N = \{p_1, \ldots, p_n\}$, and let X a team over N. We define the following formula:

$$\Theta_X := \bigvee_{v \in X} (p_1^v \wedge \cdots \wedge p_n^v),$$

whereby p_i^v stands for p_i if $v(p_1) = 1$ holds and for $\neg p_i$ if $v(p_i) = 0$ holds. This formula is of crucial importance for proving Theorem 18. It is straightforward to check the following lemma.

Lemma 19. *Θ_X defines the family of subteams of X, i.e., we have*

$$Y \models \Theta_X \iff Y \subseteq X.$$

The next lemma guarantees that for a sufficient large enough teams X exist formulas A, B such that X is a model of the disjunction $A \vee B$, but X is not a model of A and B. We make use of the following notions: define $\mathrm{down}(X) = \{Y \mid Y \subseteq X\}$ and $\mathrm{down}(X_1, \ldots, X_n) = \mathrm{down}(\{X_1, \ldots, X_n\}) = \bigcup_{i=1}^n \mathrm{down}(X_i)$

Lemma 20 (†)**.** *For each team X with $|X| > 1$ exists formulas A and B such that*

$$\begin{aligned} &X \models A \vee B\,, \\ &X \not\models A\,, \text{ and} \\ &X \not\models B \end{aligned}$$

Proof. Since we have $|X| > 1$, there exists non-empty $Y, Z \subseteq X$ teams such that $X = Y \cup Z$ and $Y \neq X$ and $Z \neq X$. Moreover, there are formulas A and B such that $[\![A]\!] = \mathrm{down}(Y)$ and $[\![B]\!] = \mathrm{down}(Z)$, namely $A = \Theta_Y$ and $B = \Theta_Z$. □

We will now show that the ($\triangle$)-property and the ($\star$)-property describe the same preferential models.

Lemma 21. *Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ be a preferential model over $\mathsf{PL}(=\!(,))$. The preferential entailment relation $\mathrel{|\!\sim}_{\mathbb{W}}$ over $\mathsf{PL}(=\!(,))$ satisfies ($\triangle$) if and only if ($\star$) is satisfied.*

Proof. Assume ($\triangle$) holds. Then it is easy to see that the minimal elements of the order $\prec$ are states that are mapped, via ℓ, to singleton teams. Furthermore, by the downward closure property, for any $A \vee B$ the minimal teams satisfying the formula are all singletons. Since for singleton teams the

[2] For the sake of readability we abuse notation and identify s_X with X.
[3] Abbreviation: $\min([\![A]\!], \prec) = \{\ell(s) \mid s \in \min(S(A), \prec)\}$

interpretation of $\vee$ is equivalent with that of the Boolean disjunction the property ($\star$) follows.

For the converse, assume that ($\star$) holds and let X be a team with $|X| > 1$. We will show that then there is some team Y with

$$Y \subsetneq X\,,\quad Y \neq \emptyset\,,\text{ and}\quad Y \prec X$$

Because X contains at least two valuations, there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$ and $Y \neq X$ and $Z \neq X$. By (the proof of) Lemma 20 there are formulas $A = \Theta_Y$ and $B = \Theta_Z$ such that $X \models A \vee B$, yet $X \not\models A$ and $X \not\models B$. Using this and ($\star$), we obtain that $X \notin \min(A \vee B, \prec)$ holds. However, by smoothness of $\prec$, the set $Mod(A \vee B) = \mathcal{P}(X)$ contains a team X' such that $X' \prec X$. Now X' is a witness for the ($\triangle$)-Property. □

Now we are ready to give the proof of Theorem 18.

Proof of Theorem 18. By Lemma 21, it suffices to show ($\star$) $\Rightarrow$ (Or) and (Or) $\Rightarrow$ ($\triangle$). We show each direction independently:

($\star$) $\Rightarrow$ (Or). This is given by Proposition 17.

(Or) $\Rightarrow$ ($\triangle$). Assume, for a contradiction, that ($\triangle$) fails. Then there exists a team X of size $j \geq 2$ such that for all $Y \subseteq X, Y \not\prec X$. Let $j = l + k$ ($l, k \geq 1$ and $l \leq k$) and define

$$\alpha := \Theta_X \wedge (\theta \vee \cdots \vee \theta),$$

where $\theta := \bigwedge_{1 \leq i \leq n} {=}(p_i)$ and α has l many copies of θ. It is easy to check that α is satisfied by subteams of X of cardinality at most l. The formula β is defined similarly with k copies of θ in the disjuncts. Now it holds that $\beta \models \beta$, $\alpha \models \beta$ but $X \not\models \alpha, \beta$. Using reflexivity and right weakening, it follows that $\beta \mid\!\sim_{\mathbb{W}} \beta$ and $\alpha \mid\!\sim_{\mathbb{W}} \beta$. On the other hand, since X is now a minimal model of $\alpha \vee \beta$ that does not satisfy β we have shown $\alpha \vee \beta \not\mid\!\sim_{\mathbb{W}} \beta$ and that (Or) fails for $\mid\!\sim_{\mathbb{W}}$. □

6.2. Relation to Dependence Logic and Classical Entailment

Theorem 18 and the $\triangle$-property imply that preferential dependence logics that satisfy System P are quintessentially the same as their flattening[4] counterpart in (preferential) propositional logic with classical (non-team) semantics.

Theorem 22. *Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ be a preferential model over* $\mathsf{PL}(=(,))$ *that satisfies System P. Then $A \mid\!\sim_{\mathbb{W}} B$ iff $A^f \mid\!\sim_{\mathbb{W}'} B^f$, where $\mathbb{W}' = \langle \mathcal{S}', \ell', \prec' \rangle$ denotes the preferential model for classical propositional logic induced by $\mathbb{W}$, i.e., over $\models^c$ for* PL *formulas and valuations induced by the singleton teams in W.*

Proof. Note first that by the assumption for all valuations s, s' it holds that $s \prec' s'$ iff $\{s\} \prec \{s'\}$. By theorem 18, W satisfies the ($\triangle$)-property and hence the minimal elements of $\prec$ are singleton teams. Hence $A \mid\!\sim B$, iff, for all minimal $\{s\} \in [\![A]\!] : \{s\} \models B$, iff, for all $\prec'$-minimal $s \in [\![A^f]\!] : s \models B^f$. The last equivalence holds due to (1). □

[4]Note that the flatting of a formula is defined at the end of Section 2.

As a last result, we consider preferential models that characterize the $\models$ entailment relation, as well as the entailment relation for classical formulas $\models^c$. Let $\mathbb{W}_{\text{sub}} = \langle \mathcal{S}_{\text{sub}}, \ell_{\text{sub}}, \prec_{\text{sub}} \rangle$ and $\mathbb{W}_{\text{sup}} = \langle \mathcal{S}_{\text{sup}}, \ell_{\text{sup}}, \prec_{\text{sup}} \rangle$ be the preferential models such that the following holds:

$$\mathcal{S}_{\text{sub}} = \mathcal{S}_{\text{sup}} = \{s_X \mid X \text{ is a non-empty team}\}$$
$$\ell_{\text{sub}}(s_X) = \ell_{\text{sup}}(s_X) = X$$
$$Y \prec_{\text{sub}} X \text{ if } Y \subsetneq X \qquad Y \prec_{\text{sup}} X \text{ if } X \subsetneq Y$$

In $\mathbb{W}_{\text{sub}}$ and $\mathbb{W}_{\text{sup}}$, for each team X there is exactly one state s_X that is labelled by X. In $\prec_{\text{sub}}$, subsets of a team are preferred, whereas in $\prec_{\text{sup}}$ superset teams are preferred.

The preferential model $\mathbb{W}_{\text{sup}}$ gives rise to the $\mathsf{PL}(=(,))$ entailment relation $\models$, and the preferential model $\mathbb{W}_{\text{sup}}$ gives rise to classical entailment of the flattening $\models^c$.

Proposition 23. *For all* $\mathsf{PL}(=(,))$*-formulas A, B we have:*

(1) $A \mid\!\sim_{\mathbb{W}_{sub}} B$ if and only if $A^f \models^c B^f$

(2) $A \mid\!\sim_{\mathbb{W}_{sup}} B$ if and only if $A \models B$

Proof. We show statements (1) and (2).

(1) Observe at first that we have $A \mid\!\sim_{\mathbb{W}_{\text{sub}}} B$ exactly when we also have $\min([\![A]\!], \prec_{\text{sub}}) \subseteq [\![B]\!]$. Because $\mathsf{PL}(=(,))$ has the downwards closure property, we also have that stating $\min([\![A]\!], \prec_{\text{sub}}) \subseteq [\![B]\!]$ is equivalent to stating that for all singleton teams $\{v\}$ holds that $\{v\} \models A$ implies $\{v\} \models B$. The latter statement is equivalent to stating that for the flattening A^f and B^f holds that for all valuations v holds that $v \models A^f$ implies $v \models B^f$ (see also Section 2). Hence, we have $A \mid\!\sim_{\mathbb{W}_{\text{sub}}} B$ if and only if $A^f \models^c B^f$.

(2) We obtain $\models \subseteq \mid\!\sim_{\mathbb{W}_{\text{sup}}}$ immediately by the definition of $\mid\!\sim_{\mathbb{W}_{\text{sup}}}$. We consider the other direction. The statement $A \models B$ is equivalent to $[\![A]\!] \subseteq [\![B]\!]$. Because $[\![A]\!]$ is downward-closed, there are (pairwise $\subseteq$-incomparable) teams $X_1, \ldots, X_n$ such that $[\![A]\!] = \mathrm{down}(X_1, \ldots, X_n)$. Because of the last property, we have that $A \models B$ holds exactly when $\{X_1, \ldots, X_n\} \subseteq [\![B]\!]$ holds. By construction of $\mathbb{W}_{\text{sup}}$ we have $\min([\![A]\!], \prec_{\text{sup}}) = \{X_1, \ldots, X_n\}$ for A. Consequently, we also have that $A \mid\!\sim_{\mathbb{W}_{\text{sup}}} B$ holds and consequently, we also have $\mid\!\sim_{\mathbb{W}_{\text{sup}}} \subseteq \models$. □

Note that, in conformance with Theorem 18 and Proposition 13, $\mathbb{W}_{\text{sup}}$ violates the ($\triangle$)-property and ($\star$)-property.

7. Conclusion

We considered preferential propositional team logics, which are non-monotonic logics in the style of Kraus et al.. Our results are a primer for further investigations on non-monotonic team logics. We want to highlight that Theorem 22 indicates that (Or) of System P is too restrictive for non-monotonic team logics. In future work, the authors plan to identify further results on preferential models, especially with respect to axiomatic systems different from System P. Connected with that is to study the meaning of conditionals and related complexity issues in the setting of team logics.

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